SENIOR SECONDARY COURSE

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MATHEMATICS

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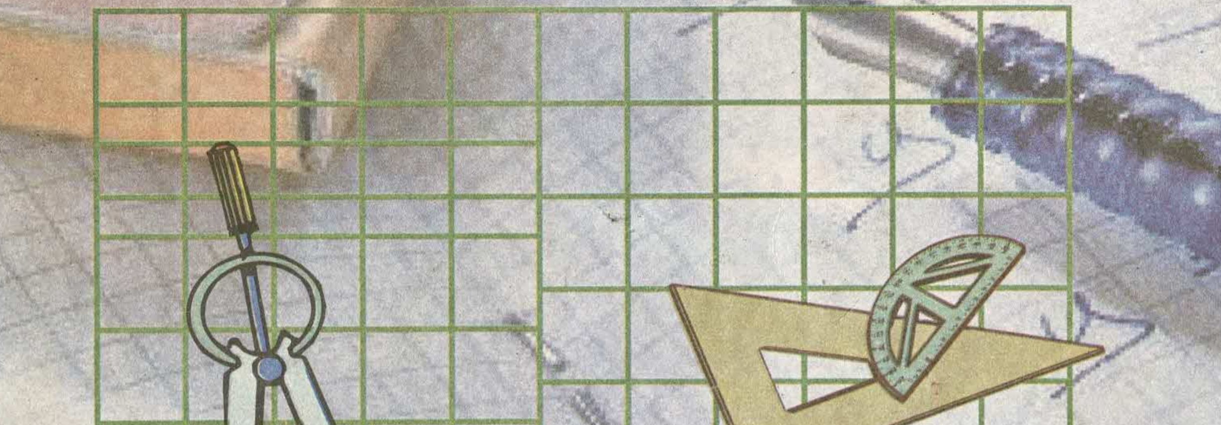

NATIONAL INSTITUTE OF OPEN SCHOOLING

SENIOR SECONDARY COURSE
MATHEMATICS

3

(OPTIONAL MODULES)

Designed and developed
by
Suvendu Sekhar Das

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विद्याधनम् सर्वधनं प्रधानम्

NATIONAL INSTITUTE OF OPEN SCHOOLING
A-24/25, INSTITUTIONAL AREA
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FOREWORD

Dear Learner,

We are very happy that you have joined NIOS and decided to become an Open and Distant Learner.

NIOS has brought out its revised Senior Secondary Course material. This course has been designed in a **modular format** in the sense that the content is divided into different modules. These modules are made up of a number of lessons. The modules are self contained and you can pick up any of the modules first that interests you. However, we would like you to proceed as the modules have been arranged because there are references and cross references to other lessons.

One of the main features of this course is the division of the modules into **core** and **optional** modules. The core modules are compulsory for all learners, but you can choose any **one** of the two optional modules. For example, in Geography, you can pick up either Local Area Planning **or** Tourism Geography; in Business Studies, you can study either Wage Employment in Business **or** Self Employment in Business. Each subject offers two optional modules. The idea is to allow you to study what interests you more even in one particular subject. This is something unique to the NIOS courses of study. You will not find such **choice** elsewhere.

This course is revised on the lines of the guidelines contained in the National Curriculum Framework 2005. We have tried to make the material as **activity based** as possible. We believe that you learn more when you do something with your own hands rather than just passive reading. We have made efforts to keep the **language of the study material simple** to facilitate you to understand the content easily.

The examples that we have chosen and used in the course material are from daily life to enable you to relate easily what is new to what you already know. Through the Self Learning Materials, we are trying to help you to construct your own knowledge so that you learn by understanding and not just by memorising everything.

Another important feature of this learning material is integration of Adolescence Education issues with the learning content. Realising that development of life skills like self awareness, critical thinking, negotiation and communication skills is important, we have used different opportunities to build desirable skills in the lessons.

The study materials developed by NIOS are **self-learning materials**. You are supposed to read and work on your own. Unlike a textbook, you do not need a teacher to tell you what to do.

With our good wishes, start studying, do what you are told to do, attempt all activities, answer the intext questions, check your answers from the answers given, learn each topic well and be a **successful self learner**.

Chairman, NIOS

A NOTE FROM THE DIRECTOR

Dear Learner,

The Academic Department at the National Institute of Open Schooling tries to bring you new programmes in accordance with your needs and requirements. After making a comprehensive study, we found that our curriculum is more functional, related to life situations and simple. The task now was to make it more effective and useful for you. We invited leading educationists of the country and under their guidance, we have been able to revise and update the Senior Secondary curriculum in the subject of Mathematics in the light of National Curriculum Framework (NCF)-2005.

At the same time, we have also removed old, outdated information and added new, relevant things and tried to make the learning material attractive and appealing for you.

I hope you will find the new material interesting and exciting . Any suggestions for further improvement are welcome.

Let me wish you all a happy and successful future.

A Word With You

Dear Learner,

In this book you will be introduced to a optional module of your course titled **"Vectors and Three Dimensional Geometry"**. In your daily life situation, you are acquainted with physical quantities like distance, speed, temperature etc., which are expressed in terms of numbers only. You have also come across physical quantities such as velocity, acceleration etc. where you need to give their magnitude and direction to express them completely. We will be dealing with such physical quantities and their properties in the first lesson of this book.

Next we move on to Three Dimensional Geometry which is the extension of Two Dimensional Plane Geometry. We have discussed the location of a point in plane Geometry in the core modules. In Three Dimensional Geometry, we locate a point in space (i.e. in Three Dimensions) by describing its perpendicular distance from three mutually perpendicular planes. Our basic aim in this module is to acquaint you with some basic facts about points, image/projection of a point, planes, angles between two planes and lines in three dimensional space.

Lastly, we have discussed the sphere- how it is different from a circle in Plane Geometry. We have also discussed the plane section of a sphere in this module. We have explained all these concepts through examples in a very simple way.

This module also contains a "**Questions For Practice**" page for self-check. Try to solve it and get it checked by the subject teacher at your study centre.

If you face any difficulty, please do not hesitate to write to us.

Your suggestions are also welcome.

Wishing you all the best.

Yours

(Suvendu Sekhar Das)
Academic Officer (Mathematics)

An Overview of the Learning Material

CORE MODULES

ALGEBRA

Complex Numbers
Quadratic Equations
Matrices
Determinants and their Applications
Inverse of a Matrix and its Application
Linear Inequations and their Applications
Permutations and Combinations
Binomial Theorem

SEQUENCES AND SERIES

Arithmetic and Geometric Progressions
Special Types of Series

CALCULUS

Limit and Continuity
Differentiation
Differentiation of Trigonometric Functions
Differentiation of Exponential and Logarithmic Functions
Tangents and Normals
Maxima and Minima
Integration
Definite Integrals
Differential Equations

COORDINATE GEOMETRY

Cartesian System of Coordinates
Straight Lines
Circles
Conic Sections

FUNCTIONS

Sets, Relations and Functions
Trigonometric Functions-I
Trigonometric Functions-II
Inverse Trigonometric Functions
Relations between Sides and Angles of a Triangle

STATISTICS

Measures of Dispersion
Random Experiments and Events
Probability

OPTIONAL MODULES

(You have to choose any one of the following modules)

VECTORS AND THREE DIMENSIONAL GEOMETRY

Vectors
Introduction to Three Dimensional Geometry
The Plane
The Straight Line
The Sphere

MATHEMATICS FOR COMMERCE, ECONOMICS AND BUSINESS

Shares and Debentures
Index Numbers
Insurance
Indirect Taxes
Application of Calculus in Commerce and Economics

CONTENTS

OPTIONAL MODULES

(You have choose any one of the following modules)

	Title	*Page No.*
OPTIONAL-I	**VECTORS AND THREE DIMENSIONAL GEOMETRY**	**1-105**
OPTIONAL-II	**MATHEMATICS FOR COMMERCE, ECONOMICS AND BUSINESS**	**107-256**

OPTIONAL-I

VECTORS AND THREE DIMENSIONAL GEOMETRY

Lesson No.	*Title*	*Page No.*
32	Vectors	1
33	Introduction to Three Dimensional Geometry	25
34	The Plane	47
35	The Straight Line	71
36	The Sphere	87
	Questions For Practice-Vectors and Three Dimensional Geometry	104

(OPTIONAL -I)

VECTORS AND THREE DIMENSIONAL GEOMETRY

(OPTIONAL -II)

MATHEMATICS FOR COMMERCE, ECONOMICS AND BUSINESS

32

VECTORS

In day to day life situations, we deal with physical quantities such as distance, speed, temperature, volume etc. These quantities are sufficient to describe change of position, rate of change of position, body temperature or temperature of a certain place and space occupied in a confined portion respectively. We also come across physical quantities such as dispacement, velocity, acceleration, momentum etc. which are of a difficult type.

Let us consider the following situation. Let A, B, C and D be four points equidistant (say 5 km each) from a fixed point P. If you are asked to travel 5 km from the fixed point P, you may reach either A, B, C, or D. Therefore, only starting (fixed point) and distance covered are not sufficient to describe the destination. We need to specify end point (terminal point) also. This idea of terminal point from the fixed point gives rise to the need for direction.

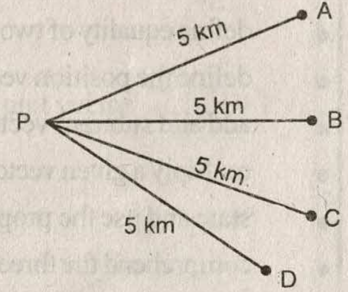


Fig. 32.1

Consider another example of a moving ball. If we wish to predict the position of the ball at any time what are the basics we must know to make such a prediction?

Let the ball be initially at a certain point A. If it were known that the ball travels in a straight line at a speed of 5cm/sec, can we predict its position after 3 seconds ? Obviously not. Perhaps we may conclude that the ball would be 15 cm away from the point A and therefore it will be at some point on the circle with A as its centre and radius 15 cms. So, the mere knowledge of speed and time taken are not sufficient to predict the position of the ball. However, if we know that the ball moves in a direction due east from A at a speed of 5cm/sec., then we shall be able to say that after 3 seconds, the ball must be precisely at the point P which is 15 cms in the direction east of A.

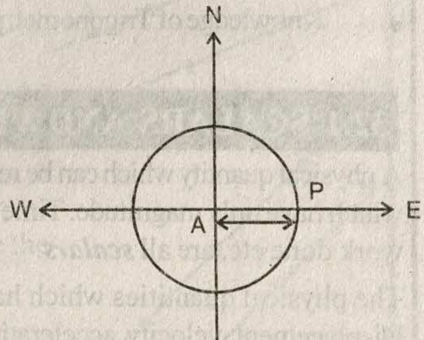


Fig. 32.2

Thus, to study the displacement of a ball after time t (3 seconds), we need to know the magnitude of its speed (i.e. 5 cm/sec) and also its direction (east of A)

In this lesson we will be dealing with quantities which have magnitude only, called scalars and the quantities which have both magnitude and direction, called vectors. We will represent vectors as directed line segments and determine their magnitudes and directions. We will study about various types of vectors and perform operations on vectors with properties thereof. We will also acquaint ourselves with position vector of a point w.r.t. some origin of reference. We will find out the resolved parts of a vector, in two and three dimensions, along two and three mutually perpendicular directions respectively. We will also derive section formula and apply that to problems. We will also define scalar and vector products of two vectors.

OBJECTIVES

After studying this lesson, you will be able to :

- explain the need of mentioning direction;
- define a scalar and a vector;
- distinguish between scalar and vactor;
- represent vectors as directed line segment;
- determine the magnitude and direction of a vector;
- classify different types of vectors-null and unit vectors;
- define equality of two vectors;
- define the position vector of a point;
- add and subtract vectors;
- multiply a given vector by a scalar;
- state and use the properties of various operations on vectors;
- comprehend the three dimensional space;
- resolve a vector along two or three mutually prependicular axes,
- derive and use section formula; and
- define scalar (dot) and vector (cross) product of two vectors.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of plane and coordinate geometry.
- Knowledge of Trigonometry.

32.1 SCALARS AND VECTORS

A physical quantity which can be represented by a number only is known as a scalar i.e, quantities which have only magnitude. Time, mass, length, speed, temperature, volume, quantity of heat, work done etc. are all ***scalars***.

The physical quantities which have magnitude as well as direction are known as vectors. Displacement, velocity, acceleration, force, weight etc. are all examples of ***vectors***.

32.2 VECTOR AS A DIRECTED LINE SEGMENT

You may recall that a line segment is a portion of a given line with two end points. Take any line

l (called a support). The portion of L with end points A and B is called a line segment. The line segment AB along with direction from A to B is written as $\overrightarrow{AB}$ and is called a directed line segment. A and B are respectively called the initial point and terminal point of the vector $\overrightarrow{AB}$.

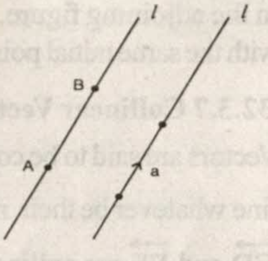


Fig. 32.3

The length AB is called the ***magnitude*** or ***modulus*** of $\overrightarrow{AB}$ and is denoted by $|\overrightarrow{AB}|$. In other words the length $AB = |\overrightarrow{AB}|$.

Scalars are usually represented by a, b, c etc. whereas vectors are usually denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$ etc. Magnitude of a vector $\vec{a}$ i.e., $|\vec{a}|$ is usually denoted by 'a'.

32.3 CLASSIFICATION OF VECTORS

32.3.1 Zero Vector (Null Vector)

A vector whose magnitude is zero is called a zero vector or ***null vector***. Zero vector has not definite direction. $\overrightarrow{AA}$, $\overrightarrow{BB}$ are zero vectors. Zero vectors is also denoted by $\vec{0}$ to distinguish it from the scalar 0.

32.3.2 Unit Vector

A vector whose magnitude is unity is called a ***unit vector***. So for a unit vector $\vec{a}$, $|\vec{a}| = 1$. A unit vector is usually denoted by $\hat{a}$. Thus, $\vec{a} = |\vec{a}|\,\hat{a}$.

32.3.3 Equal Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude. i.e., $|\vec{a}| = |\vec{b}|$ and the same direction as shown in Fig. 32.4. Symbolically, it is denoted by $\vec{a} = \vec{b}$.

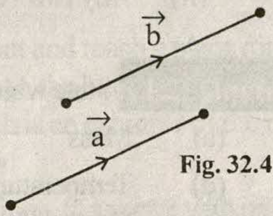


Fig. 32.4

Remark : Two vectors may be equal even if they have different parallel lines of support.

32.3.4 Like Vectors

Vectors are said to be like if they have same direction whatever be their magnitudes. In the adjoining Fig. 32.5, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are like vectors, although their magnitudes are not same.

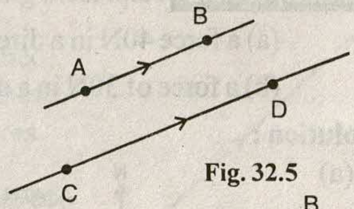


Fig. 32.5

32.3.5 Negative of a Vector

$\overrightarrow{BA}$ is called the ***negative of the vector*** $\overrightarrow{AB}$, when they have the same magnitude but opposite directions.

i.e. $\overrightarrow{BA} = -\overrightarrow{AB}$

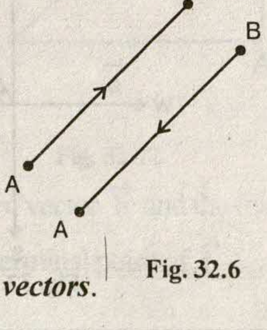


Fig. 32.6

32.3.6 Co-initial Vectors

Two or more vectors having the same initial point are called ***Co-initial vectors***.

In the adjoining figure, $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AC}$ are co-initial vectors with the same initial point A.

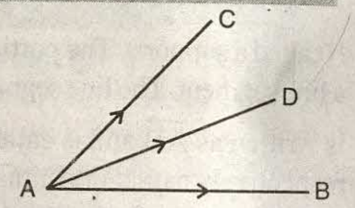


Fig. 32.7

32.3.7 Collinear Vectors

Vectors are said to be collinear when they are parallel to the same line whatever be their magnitudes. In the adjoining figure, $\overrightarrow{AB}$, $\overrightarrow{CD}$ and $\overrightarrow{EF}$ are collinear vectors. $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are also ***collinear***.

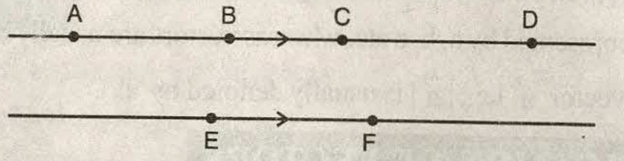


Fig. 32.8

32.3.8 Co-planar Vectors

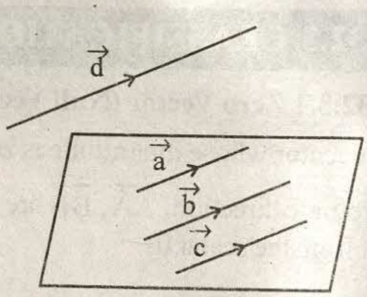


Fig. 32.9

Vectors are said to be co-planar when they are parallel to the same plane. In the adjoining figure $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are co-planar. Whereas $\vec{a}$, $\vec{b}$ and $\vec{c}$ lie on the same plane, $\vec{d}$ is parallel to the plane of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Note : (i) A zero vector can be made to be collinear with any vector.

(ii) Any two vectors are always co-planar.

Example 32.1 State which of the following are scalars and which are vectors. Give reasons.

(a)	Mass	(b)	Weight	(c)	Momentum
(d)	Temperature	(e)	Force	(f)	Density

Solution : (a), (d) and (f) are scalars because these have only magnitude while (b), (c) and (e) are vectors as these have magnitude and direction as well.

Example 32.2 Represent graphically

(a) a force 40N in a direction 60° north of east.

(b) a force of 30N in a direction 40° east of north.

Solution :

(a)

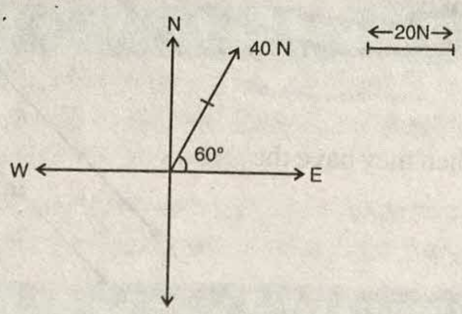


Fig. 32.10

(b)

2

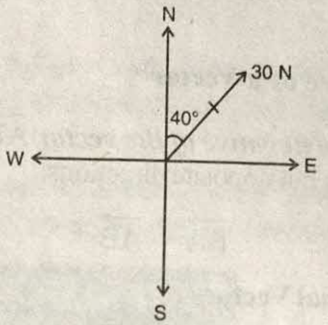


Fig. 32.11

CHECK YOUR PROGRESS 32.1

1. Which of the following is a scalar quantity ?

   (a) Displacement (b) Velocity (c) Force (d) Length.

2. Which of the following is a vector quantity ?

   (a) Mass (b) force (c) time (d) tempertaure

3. You are given a displacement vector of 5 cm due east. Show by a diagram the corresponding negative vector.

4. Distinguish between like and equal vectors.

5. Represent graphically

   (a) a force 60 Newton is a direction 60° west of north.

   (b) a force 100 Newton in a direction 45° north of west.

32.4 ADDITION OF VECTORS

Recall that you have learnt four fundamental operations viz. addition, subtraction, multiplication and division on numbers. The addition (subtraction) of vectors is different from that of numbers (scalars).

In fact, there is the concept of resultant of two vectors (these could be two velocities, two forces etc.) We illustrate this with the help of the following example :

Let us take the case of a boat-man trying to cross a river in a boat and reach a place directly in the line of start. Even if he starts in a direction perpendicular to the bank, the water current carries him to a place different from the place he desired., which is an example of the effect of two velocities resulting in a third one called the resultant velocity.

Thus, two vectors with magnitudes 3 and 4 may not result, on addition, in a vector with magnitude 7. It will depend on the direction of the two vectors i.e., on the angle between them. The addition of vectors is done in accordance with the triangle law of addition of vectors.

32.4.1 Triangle Law of Addition of Vectors

A vector whose effect is equal to the resultant (or combined) effect of two vectors is defined as the resultant or sum of these vectors. This is done by the triangle law of addition of vectors.

In the adjoining Fig. 32.12 vector $\overrightarrow{OB}$ is the resultant or sum of vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ and is written as

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

i.e. $$\vec{a} + \vec{b} = \overrightarrow{OB} = \vec{c}$$

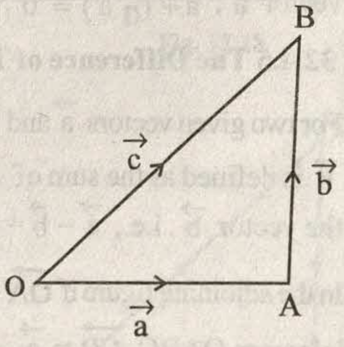


Fig. 32.12

You may note that the terminal point of vector $\vec{a}$ is the initial point of vector $\vec{b}$ and the initial point of $\vec{a} + \vec{b}$ is the initial point of $\vec{a}$ and its terminal point is the terminal point of $\vec{b}$.

32.4.2 Addition of more than two Vectors

Addition of more then two vectors is shown in the adjoining figure

$$\vec{a}+\vec{b}+\vec{c}+\vec{d}$$
$$=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}$$
$$=\overrightarrow{AC}+\overrightarrow{CD}+\overrightarrow{DE}$$
$$=\overrightarrow{AD}+\overrightarrow{DE}$$
$$=\overrightarrow{AE}$$

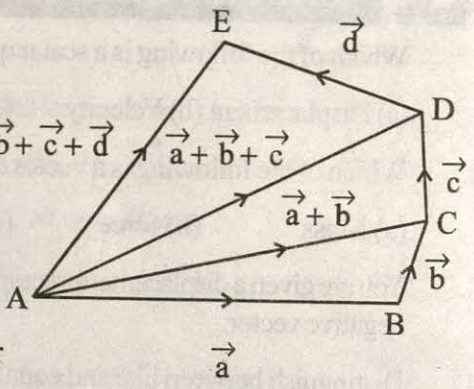


Fig. 32.13

The vector $\overrightarrow{AE}$ is called the sum or the resultant vector of the given vectors.

32.4.3 Parallelogram Law of Addition of Vectors

Recall that two vectors are equal when their magnitude and direction are the same. But they could be parallel [refer to Fig. 32.14].

See the parallelogram OABC in the adjoining figure :

We have, $\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$

But $\overrightarrow{AB}=\overrightarrow{OC}$

$\therefore$ $\overrightarrow{OA}+\overrightarrow{OC}=\overrightarrow{OB}$

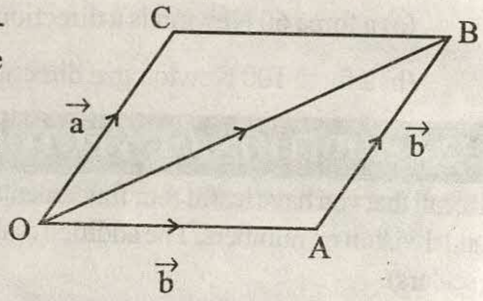


Fig. 32.14

which is the parallelogram law of addition of vectors. **If two vectors are represented by the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal through the common point of the adjacent sides.**

32.4.4 Negative of a Vector

For any vector $\vec{a}=\overrightarrow{OA}$, the negative of $\vec{a}$ is represented by $\overrightarrow{AO}$. The negative of $\overrightarrow{AO}$ is the same as $\overrightarrow{OA}$. Thus, $|\overrightarrow{OA}|=|\overrightarrow{AO}|=|\vec{a}|$ and $\overrightarrow{OA}=-\overrightarrow{AO}$. It follows from definition that for any vector $\vec{a}$, $\vec{a}+(-\vec{a})=\vec{0}$.

32.4.5 The Difference of Two Given Vectors

For two given vectors $\vec{a}$ and $\vec{b}$, the difference $\vec{a}-\vec{b}$ is defined as the sum of $\vec{a}$ and the negative of the vector $\vec{b}$. i.e., $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$.

In the adjoining figure if $\overrightarrow{OA}=\vec{a}$ then, in the parallelogram OABC, $\overrightarrow{CB}=\vec{a}$

and $\overrightarrow{BA}=-\vec{b}$.

$\therefore$ $\overrightarrow{CA}=\overrightarrow{CB}+\overrightarrow{BA}=\vec{a}-\vec{b}$

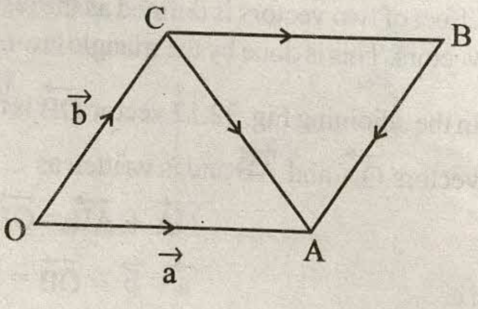


Fig. 32.15

Example 32.3 When is the sum of two non-zero vectors zero ?

Solution : The sum of two non-zero vectors is zero when they have the same magnitude but opposite direction.

Example 32.4 Show by a diagram $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

Solution : From the adjoining figure, resultant

$$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}$$

$$=\vec{a}+\vec{b} \qquad \text{....(i)}$$

Complete the parallelogram OABC

$$\overrightarrow{OC}=\overrightarrow{AB}=\vec{b},\ \overrightarrow{CB}=\overrightarrow{OA}=\vec{a}$$

$$\therefore \quad \overrightarrow{OB}=\overrightarrow{OC}+\overrightarrow{CB}$$

$$=\vec{b}+\vec{a} \qquad \text{.....(ii)}$$

$$\therefore \quad \vec{a}+\vec{b}=\vec{b}+\vec{a} \qquad \text{[From (i) and (ii)]}$$

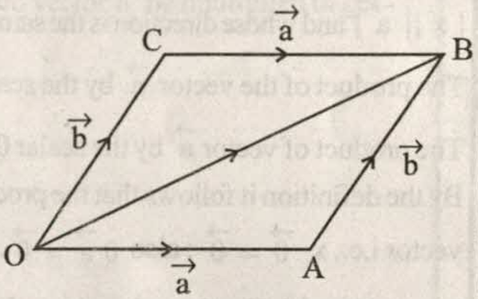


Fig. 32.16

CHECK YOUR PROGRESS 32.2

1. The diagonals of the parallelogram ABCD intersect at the point O. Find the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$.

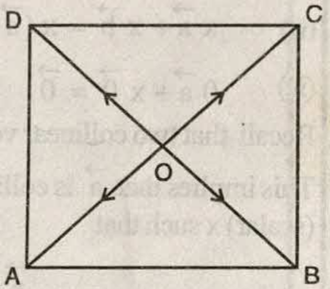


Fig. 32.17

2. The medians of the triangle ABC intersect at the point O. Find the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

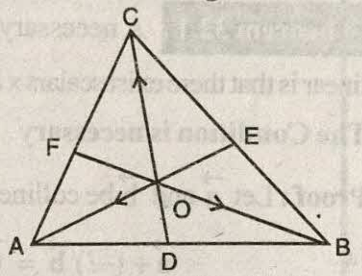


Fig. 32.18

32.5 POSITION VECTOR OF A POINT

We fix an arbitrary point O in space. Given any point P in space, we join it to O to get the vector $\overrightarrow{OP}$. This is called the position vector of the point P with respect to O, called the ***origin of reference***. Thus, to each given point in space there corresponds a unique position vector with respect to a given origin of reference. Conversely, given an origin of reference O, to each vector with the initial point O, corresponds a point namely, its terminal point in space.

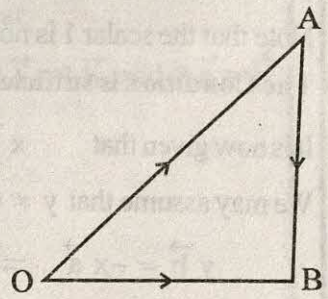


Fig. 32.19

Consider a vector AB. Let O be the origin of reference.

Then $\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$ or $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

or $\overrightarrow{AB}$ = (Position vector of terminal point B)–(Position vector of initial point A)

32.6 MULTIPLICATION OF A VECTOR BY A SCALAR

The product of a non-zero vector $\vec{a}$ by the scalar $x \neq 0$ is a vector whose length is equal to $|x||\vec{a}|$ and whose direction is the same as that of $\vec{a}$ if $x > 0$ and opposite to that of $\vec{a}$ if $x < 0$.

The product of the vector $\vec{a}$ by the scalar x is denoted by $x\vec{a}$.

The product of vector $\vec{a}$ by the scalar 0 is the vector $\vec{0}$.

By the definition it follows that the product of a zero vector by any non-zero scalar is the zero vector i.e., $x\vec{0} = \vec{0}$; also $0\vec{a} = \vec{0}$.

Laws of multiplication of vectors : If $\vec{a}$ and $\vec{b}$ are vectors and x, y are scalars, then

(i) $x(y\vec{a}) = (x\,y)\vec{a}$

(ii) $x\vec{a} + y\vec{a} = (x + y)\vec{a}$

(iii) $x\vec{a} + x\vec{b} = x(\vec{a} + \vec{b})$

(iv) $0\vec{a} + x\vec{0} = \vec{0}$

Recall that two collinear vectors have the same direction but may have different magnitudes. This implies that $\vec{a}$ is collinear with a non-zero vector $\vec{b}$ if and only if there exists a number (scalar) x such that

$$\vec{a} = x\vec{b}$$

Theorem 32.1 A necessary and sufficient condition for two vectors $\vec{a}$ and $\vec{b}$ to be collinear is that there exist scalars x and y (not both zero simultaneously) such that $x\vec{a} + y\vec{b} = \vec{0}$.

The Condition is necessary

Proof : Let $\vec{a}$ and $\vec{b}$ be collinear. Then there exists a scalar *l* such that $\vec{a} = l\vec{b}$

i.e., $\vec{a} + (-l)\vec{b} = \vec{0}$

$\therefore$ We are able to find scalars $x\,(=1)$ and $y\,(=-l)$ such that $x\vec{a} + y\vec{b} = \vec{0}$

Note that the scalar 1 is non-zero.

The Condition is sufficient

It is now given that $x\vec{a} + y\vec{b} = \vec{0}$ and $x \neq 0$ and $y \neq 0$ simultaneously.

We may assume that $y \neq 0$

$\therefore$ $y\vec{b} = -x\vec{a} \Rightarrow \vec{b} = -\frac{x}{y}\vec{a}$ i.e., $\vec{b}$ and $\vec{a}$ are collinear.

Corollary : Two vectors $\vec{a}$ and $\vec{b}$ are non-collinear if and only if every relation of the form $x\vec{a} + y\vec{b} = \vec{0}$ given as $x = 0$ and $y = 0$.

[**Hint :** If any one of x and y is non-zero say y, then we get $\vec{b} = -\frac{x}{y}\vec{a}$ which is a contradiction]

Example 32.5 Find the number x by which the non-zero vector $\vec{a}$ be multiplied to get

(i) $\hat{a}$ (ii) $-\hat{a}$

Solution : (i) $x\vec{a} = \hat{a}$ i.e., $x|\vec{a}|\hat{a} = \hat{a}$

$$\Rightarrow \quad x = \frac{1}{|\vec{a}|}$$

(ii) $x\vec{a} = -\hat{a}$ i.e., $x|\vec{a}|\hat{a} = -\hat{a}$

$$\Rightarrow \quad x = -\frac{1}{|\vec{a}|}$$

Example 32.6 The vectors $\vec{a}$ and $\vec{b}$ are not collinear. Find x such that the vector

$$\vec{c} = (x-2)\vec{a} + \vec{b} \text{ and } \vec{d} = (2x+1)\vec{a} - \vec{b}$$

Solution : $\vec{c}$ is non-zero since the co-efficient of $\vec{b}$ is non-zero.

$\therefore$ There exists a number y such that $\vec{d} = y\vec{c}$

i.e. $(2x+1)\vec{a} - \vec{b} = y(x-2)\vec{a} + y\vec{b}$

$\therefore$ $(yx - 2y - 2x - 1)\vec{a} + (y+1)\vec{b} = 0$

As $\vec{a}$ and $\vec{b}$ are non-collinear.

$\therefore$ $yx - 2y - 2x - 1 = 0$ and $y + 1 = 0$

Solving these we get $y = -1$ and $x = \frac{1}{3}$

Thus $\vec{c} = -\frac{5}{3}\vec{a} + \vec{b}$ and $\vec{d} = \frac{5}{3}\vec{a} - \vec{b}$

We can see that $\vec{c}$ and $\vec{d}$ are opposite vectors and hence are collinear.

Example 32.7 The position vectors of two points A and B are $2\vec{a} + 3\vec{b}$ and $3\vec{a} + \vec{b}$ respectively. Find $\overrightarrow{AB}$.

Solution : Let O be the origin of reference.

Then $\overrightarrow{AB}$ = Position vector of B — Position vector of A

$$= \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (3\vec{a} + \vec{b}) - (2\vec{a} + 3\vec{b})$$

$$= (3-2)\vec{a} + (1-3)\vec{b} = \vec{a} - 2\vec{b}$$

Example 32.8 Show that the points P, Q and R with position vectors $\vec{a} - 2\vec{b}$, $2\vec{a} + 3\vec{b}$ and $-7\vec{b}$ respectively are collinear.

Solution : $\overrightarrow{PQ}$ = Position vector of Q — Position vector of P

$$= (2\vec{a} + 3\vec{b}) - (\vec{a} - 2\vec{b})$$

$$= \vec{a} + 5\vec{b} \quad \text{....(i)}$$

and $\overrightarrow{QR}$ = Position vector of R — Position vector of Q

$$= -7\vec{b} - (2\vec{a} + 3\vec{b})$$

$$= -7\vec{b} - 2\vec{a} - 3\vec{b}$$

$$= -2\vec{a} - 10\vec{b}$$

$$= -2(\vec{a} + 5\vec{b}) \quad \text{....(ii)}$$

From (i) and (ii) we get $\overrightarrow{PQ} = -2\,\overrightarrow{QR}$, a scalar multiple of $\overrightarrow{QR}$

$\therefore \quad \overrightarrow{PQ} \,||\, \overrightarrow{QR}$

But Q is a common point

$\therefore \quad \overrightarrow{PQ}$ and $\overrightarrow{QR}$ are collinear. Hence points P, Q and R are collinear.

CHECK YOUR PROGRESS 32.3

1. The position vectors of the points A and B are $\vec{a}$ and $\vec{b}$ respectively with respect to a given origin of reference. Find $\overrightarrow{AB}$.
2. Interpret each of the following :

   (i) $3\vec{a}$ (ii) $-5\vec{b}$

3. The position vectors of points A, B, C and D are respectively $2\vec{a}$, $3\vec{b}$, $4\vec{a} + 3\vec{b}$ and $\vec{a} + 2\vec{b}$. Find $\overrightarrow{DB}$ and $\overrightarrow{AC}$.
4. Find the magnitude of the product of a vector $\vec{n}$ by a scalar y.
5. State whether the product of a vector by a scalar is a scalar or a vector.
6. State the condition of collinearity of two vectors $\vec{p}$ and $\vec{q}$.
7. Show that the points with position vectors $5\vec{a} + 6\vec{b}$, $7\vec{a} - 8\vec{b}$ and $3\vec{a} + 20\vec{b}$ are collinear.

32.7 CO-PLANARITY OF VECTORS

Given any two non-collinear vectors $\vec{a}$ and $\vec{b}$, they can be made to lie in one plane. There (in the plane), the vectors will be intersecting. We take their common point as O and let the two vectors be $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Given a third vector $\vec{c}$, coplanar with $\vec{a}$ and $\vec{b}$, we can choose its initial point also as O. Let C be its terminal point. With $\overrightarrow{OC}$ as diagonal complete the parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides.

$$\therefore \qquad \vec{c} = l\,\vec{a} + m\,\vec{b}$$

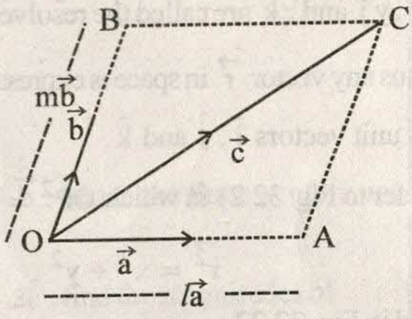


Fig. 32.20

Thus, any $\vec{c}$, coplanar with $\vec{a}$ and $\vec{b}$, is expressible as a linear combination of $\vec{a}$ and $\vec{b}$.

i.e. $\qquad \vec{c} = l\,\vec{a} + m\,\vec{b}$.

32.8 RESOLUTION OF A VECTOR ALONG TWO PERPENDICULAR AXES

Consider two mutually perpendicular unit vectors $\hat{i}$ and $\hat{j}$ along two mutually perpendicular axes OX and OY. We have seen above that any vector $\vec{r}$ in the plane of $\hat{i}$ and $\hat{j}$, can be written in the form $\vec{r} = x\hat{i} + y\hat{j}$

If O is the initial point of $\vec{r}$, then OM = x and ON = y and $\overrightarrow{OM}$ and $\overrightarrow{ON}$ are called the component vectors of $\vec{r}$ along x-axis and y-axis. $\overrightarrow{OM}$ and $\overrightarrow{ON}$, in this special case, are also called the ***resolved parts*** of $\vec{r}$

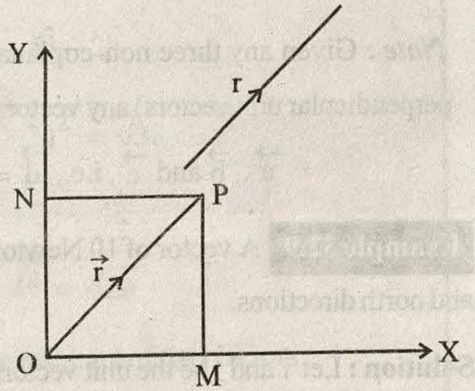


Fig. 32.21

32.9 RESOLUTION OF A VECTOR IN THREE DIMENSIONS ALONG THREE MUTUALLY PERPENDICULAR AXES

The concept of resolution of a vector in three dimensions along three mutually perpendicular axes is an extension of the resolution of a vector in a plane along two mutually perpendicular axes.

Any vector $\vec{r}$ in space can be expressed as a linear combination of three mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ as is shown in the adjoining Fig. 32.22. We complete the rectangular parallelopiped with $\overrightarrow{OP} = \vec{r}$ as its diagonal :

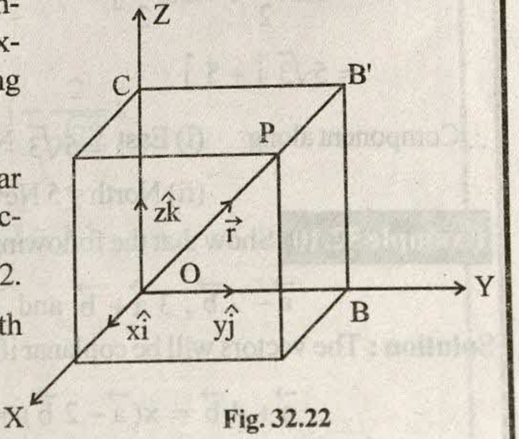


Fig. 32.22

then $\qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$x\hat{i}, y\hat{j}$ and $z\hat{k}$ are called the resolved parts of $\vec{r}$ along three mutually perpendicular axes.

Thus any vector $\vec{r}$ in space is expressible as a linear combination of three mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Refer to Fig. 32.21 in which $OP^2 = OM^2 + ON^2$ (Two dimensions)

or $$\vec{r}^2 = x^2 + y^2 \qquad \text{......(i)}$$

and in Fig. 32.22

$$OP^2 = OA^2 + OB^2 + OC^2$$

$$\vec{r}^2 = x^2 + y^2 + z^2 \qquad \text{.......(ii)}$$

Magnitude of $\vec{r} = |\vec{r}|$ in case of (i) is $\sqrt{x^2 + y^2}$

and (ii) is $\sqrt{x^2 + y^2 + z^2}$

Note : Given any three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (not necessarily mutually perpendicular unit vectors) any vector $\vec{d}$ is expressible as a linear combination of

$$\vec{a}, \vec{b} \text{ and } \vec{c}, \text{ i.e., } \vec{d} = x\vec{a} + y\vec{b} + z\vec{c}$$

Example 32.9 A vector of 10 Newton is 30° north of east. Find its components along east and north directions.

Solution : Let $\hat{i}$ and $\hat{j}$ be the unit vectors along $\overrightarrow{OX}$ and $\overrightarrow{OY}$ (East and North respectively) Resolve OP in the direction OX and OY.

$$\therefore \quad \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{ON}$$

$$= 10\cos 30°\,\hat{i} + 10\sin 30°\,\hat{j}$$

$$= 10.\frac{\sqrt{3}}{2}\hat{i} + 10.\frac{1}{2}\hat{j}$$

$$= 5\sqrt{3}\,\hat{i} + 5\,\hat{j}$$

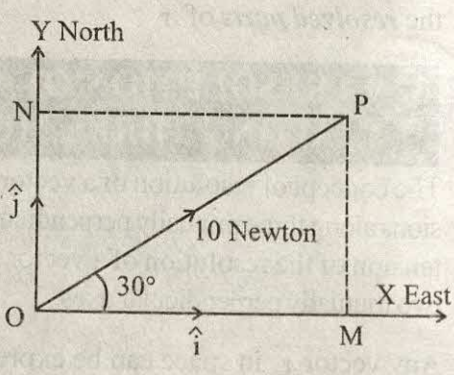


Fig. 32.23

$\therefore$ Component along (i) East $= 5\sqrt{3}$ Newton

(ii) North = 5 Newton

Example 32.10 Show that the following vectors are coplanar :

$$\vec{a} - 2\vec{b},\ 3\vec{a} + \vec{b} \text{ and } \vec{a} + 4\vec{b}$$

Solution : The vectors will be coplanar if there exists scalars x and y such that

$$\vec{a} + 4\vec{b} = x(\vec{a} - 2\vec{b}) + y(3\vec{a} + \vec{b})$$

$$= (x + 3y)\vec{a} + (-2x + y)\vec{b} \qquad \text{.....(i)}$$

Comparing the co-efficients of $\vec{a}$ and $\vec{b}$ on both sides of (i), we get

$$x + 3y = 1 \text{ and } -2x + y = 4$$

which on solving, gives $x = -\frac{11}{7}$ and $y = \frac{6}{7}$

As $\vec{a} + 4\vec{b}$ is expressible in terms of $\vec{a} - 2\vec{b}$ and $3\vec{a} + \vec{b}$, hence the three vectors are coplanar.

Example 32.11 Given $\vec{r_1} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{r_2} = 2\hat{i} - 4\hat{j} - 3\hat{k}$, find the magnitudes of

(a) $\vec{r_1}$ (b) $\vec{r_2}$ (c) $\vec{r_1} + \vec{r_2}$ (d) $\vec{r_1} - \vec{r_2}$

Solution :

(a) $|\vec{r_1}| = |\hat{i} - \hat{j} + \hat{k}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

(b) $|\vec{r_2}| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}$

(c) $\vec{r_1} + \vec{r_2} = (\hat{i} - \hat{j} + \hat{k}) + (2\hat{i} - 4\hat{j} - 3\hat{k}) = 3\hat{i} - 5\hat{j} - 2\hat{k}$

$\therefore$ $|\vec{r_1} + \vec{r_2}| = |3\hat{i} - 5\hat{j} - 2\hat{k}| = \sqrt{3^2 + (-5)^2 + (-2)^2} = \sqrt{38}$

(d) $\vec{r_1} - \vec{r_2} = (\hat{i} - \hat{j} + \hat{k}) - (2\hat{i} - 4\hat{j} - 3\hat{k}) = -\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore$ $|\vec{r_1} - \vec{r_2}| = |-\hat{i} + 3\hat{j} + 4\hat{k}| = \sqrt{(-1)^2 + 3^2 + 4^2} = \sqrt{26}$

Example 32.12 Determine the unit vector parallel to the resultant of two vectors

$$\vec{a} = 3\hat{i} + 2\hat{j} - 4\hat{k} \text{ and } \vec{b} = \hat{i} + \hat{j} + 2\hat{k}$$

Solution : The resultant vector $\vec{R} = \vec{a} + \vec{b} = (3\hat{i} + 2\hat{j} - 4\hat{k}) + (\hat{i} + \hat{j} + 2\hat{k})$

$$= 4\hat{i} + 3\hat{j} - 2\hat{k}$$

Magnitude of the resultant vector $\vec{R}$ is $|\vec{R}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$

$\therefore$ The unit vector parallel to the resultant vector

$$\frac{R}{|\vec{R}|} = \frac{1}{\sqrt{29}}(4\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$

Example 32.13 Find a unit vector in the direction of $\vec{r} - \vec{s}$

where $\vec{r} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{s} = 2\hat{i} - \hat{j} + 2\hat{k}$

Solution : $\vec{r} - \vec{s} = (\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} - \hat{j} + 2\hat{k})$

$$= -\hat{i} + 3\hat{j} - 5\hat{k}$$

$$\therefore \qquad |\vec{r} - \vec{s}| = \sqrt{(-1)^2 + (3)^2 + (-5)^2} = \sqrt{35}$$

$\therefore$ Unit vector in the direction of $(\vec{r} - \vec{s})$

$$= \frac{1}{\sqrt{35}}(-\hat{i} + 3\hat{j} - 5\hat{k}) = -\frac{1}{\sqrt{35}}\hat{i} + \frac{3}{\sqrt{35}}\hat{j} - \frac{5}{\sqrt{35}}\hat{k}$$

Example 32.14 Find a unit vector in the direction of $2\vec{a} + 3\vec{b}$ where $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} - \hat{k}$.

Solution : $2\vec{a} + 3\vec{b} = 2(\hat{i} + 3\hat{j} + \hat{k}) + 3(3\hat{i} - 2j - \hat{k})$

$$= (2\hat{i} + 6\hat{j} + 2\hat{k}) + (9\hat{i} - 6j - 3\hat{k})$$

$$= 11\hat{i} - \hat{k}.$$

$$\therefore \qquad |2\vec{a} + 3\vec{b}| = \sqrt{(11)^2 + (-1)^2} = \sqrt{122}$$

$\therefore$ Unit vector in the direction of $(2\vec{a} + 3\vec{b})$ is $\frac{11}{\sqrt{122}}\hat{i} - \frac{1}{\sqrt{122}}\hat{k}$.

Example 32.15 Show that the following vectors are coplanar :

$4\vec{a} - 2\vec{b} - 2\vec{c}$, $-2\vec{a} + 4\vec{b} - 2\vec{c}$ and $-2\vec{a} - 2\vec{b} + 4\vec{c}$ where $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors.

Solution : If these vectors be co-planar, it will be possible to express one of them as a linear combination of other two.

Let $\quad -2\vec{a} - 2\vec{b} + 4\vec{c} = x(4\vec{a} - 2\vec{b} - 2\vec{c}) + y(-2\vec{a} + 4\vec{b} - 2\vec{c})$

where x and y are scalars,

Comparing the co-efficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ from both sides, we get

$$4x - 2y = -2, \; -2x + 4y = -2 \text{ and } -2x - 2y = 4$$

These three equations are satisfied by $x = -1$, $y = -1$ Thus,

$$-2\vec{a} - 2\vec{b} + 4\vec{c} = (-1)(4\vec{a} - 2\vec{b} - 2\vec{c}) + (-1)(-2\vec{a} + 4\vec{b} - 2\vec{c})$$

Hence the three given vectors are co-planar.

CHECK YOUR PROGRESS 32.4

1. Write the condition that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are co-planar.
2. Determine the resultant vector $\vec{r}$ whose components along two rectangular Cartesian co-ordinate axes are 3 and 4 units respectively.

3. In the adjoining figure :

   $|OA| = 4$, $|OB| = 3$ and

   $|OC| = 5$. Express OP in terms of its component vectors.

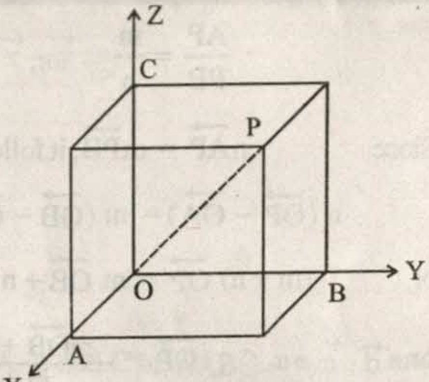


Fig. 32.24

4. If $\vec{r_1} = 4\hat{i} + \hat{j} - 4\hat{k}$, $\vec{r_2} = -2\hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{r_3} = \hat{i} + 3\hat{j} - \hat{k}$ then show that

   $|\vec{r_1} + \vec{r_2} + \vec{r_3}| = 7$.

5. Determine the unit vector parallel to the resultant of vectors :

   $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

6. Find a unit vector in the direction of vector $3\vec{a} - 2\vec{b}$ where $\vec{a} = \hat{i} - \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$.

7. Show that the following vectors are co-planar :

   $3\vec{a} - 7\vec{b} - 4\vec{c}$, $3\vec{a} - 2\vec{b} + \vec{c}$ and $\vec{a} + \vec{b} + 2\vec{c}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors.

32.10 SECTION FORMULA

Recall that the position vector of a point P is space with respect to an origin of reference O is $\vec{r} = \overrightarrow{OP}$.

In the following, we try to find the position vector of a point dividing a line segment joining two points in the ratio m : n internally.

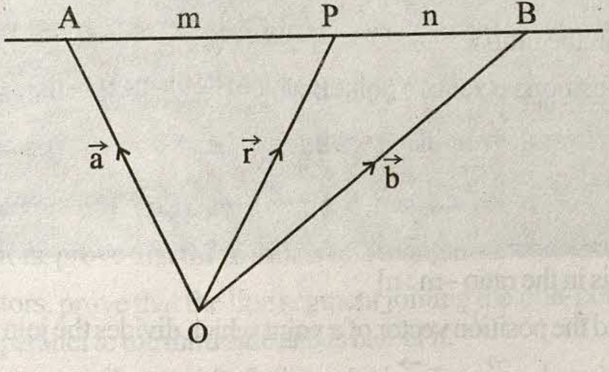


Fig. 32.25

Let A and B be two points and $\vec{a}$ and $\vec{b}$ be their position vectors w.r.t. the origin of reference O, so that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Let P divide AB in the ratio m : n so that

$$\frac{AP}{PB} = \frac{m}{n} \quad \text{or,} \quad n\overrightarrow{AP} = m\overrightarrow{PB} \qquad ....(i)$$

Since $n\overrightarrow{AP} = m\overrightarrow{PB}$, it follows that

$$n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OP})$$

or $$(m+n)\overrightarrow{OP} = m\overrightarrow{OB} + n\overrightarrow{OA}$$

or $$\overrightarrow{OP} = \frac{m\overrightarrow{OB} + n\overrightarrow{OA}}{m+n}$$

or $$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

where $\vec{r}$ is the position vector of P with respect to O.

Corollary 1 : If $\frac{m}{n} = 1 \Rightarrow m = n$, then P becomes mid-point of AB.

$\therefore$ The position vector of the mid-point of the join of two given points, whose position vectors are $\vec{a}$ and $\vec{b}$, is given by $\frac{1}{2}(\vec{a} + \vec{b})$.

Corollary 2 : The position vector P can also be written as

$$\vec{r} = \frac{\vec{a} + \frac{m}{n}\vec{b}}{1 + \frac{m}{n}} = \frac{\vec{a} + k\vec{b}}{1+k}, \qquad .....(ii)$$

where $k = \frac{m}{n}$, $k \neq -1$.

(ii) represents the position vector of a point which divides the join of two points with position vectors $\vec{a}$ and $\vec{b}$, in the ratio k : 1.

Corollary 3 : The position vector of a point P which divides AB in the ratio m : n externally is

$$\vec{r} = \frac{n\vec{a} - m\vec{b}}{n-m}$$

[**Hint :** This division is in the ratio –m : n]

Example 32.16 Find the position vector of a point which divides the join of two points whose position vectors are given by $\vec{x}$ and $\vec{y}$ in the ratio 2 : 3 internally.

Solution : Let $\vec{r}$ be the position vector of the point.

$$\therefore \quad \vec{r} = \frac{3\vec{x} + 2\vec{y}}{3+2} = \frac{1}{5}(3\vec{x} + 2\vec{y}).$$

Example 32.17 Find the position vector of mid-point of the line segment AB, if the position vectors of A and B are respectively, $\vec{x}+2\vec{y}$ and $2\vec{x}-\vec{y}$.

Solution : Position vector of mid-point of AB

$$=\frac{(\vec{x}+2\vec{y})+(2\vec{x}-\vec{y})}{2}$$

$$=\frac{3}{2}\vec{x}+\frac{1}{2}\vec{y}$$

Example 32.18 The position vectors of vertices A, B and C of ΔABC are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Find the position vector of the centroid of ΔABC.

Solution : Let D be the mid-point of side BC of ΔABC.

Let G be the centroid of ΔABC. Then G divides AD in the ratio 2 : 1 i.e. AG : GD = 2 : 1.

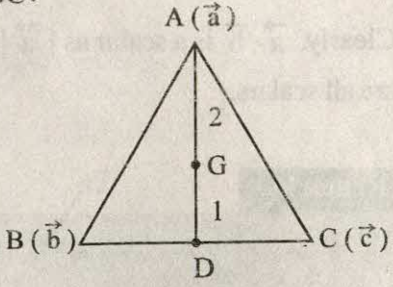


Fig. 32.26

Now position vector of D is $\dfrac{\vec{b}+\vec{c}}{2}$

$$\therefore \text{ Position vector of G is } \frac{2\cdot\dfrac{\vec{b}+\vec{c}}{2}+1\cdot\vec{a}}{2+1}$$

$$=\frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

CHECK YOUR PROGRESS 32.5

1. Find the position vector of the point C if it divides AB in the ratio (i) $\frac{1}{2} : \frac{1}{3}$

   (ii) 2 : − 3, given that the position vectors of A and B are $\vec{a}$ and $\vec{b}$ respectively.
2. Find the point which divides the join of $P(\vec{p})$ and $Q(\vec{q})$ internally in the ratio 3 : 4.
3. CD is trisected at points P and Q. Find the position vectors of points of trisection, if the position vectors of C and D are $\vec{c}$ and $\vec{d}$ respectively
4. Using vectors, prove that the medians of a triangle are concurrent.
5. Using vectors, prove that the line segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.

32.11 PRODUCT OF VECTORS

In Section 32.9, you have multiplied a vector by a scalar. The product of vector with a scalar gives us a vector quantity. In this section we shall take the case when a vector is multiplied by another vector. There are two cases :

(i) When the product of two vectors is a scalar, we call it a scalar product, also known as dot product corresponding to the symbol '•' used for this product.

(ii) When the product of two vectors is a vector, we call it a vector product, also known as cross product corresponding to the symbol '×' used for this product.

32.12 SCALAR PRODUCT OF THE VECTORS

Let $\vec{a}$ and $\vec{b}$ two vectors and θ be the angle between them. The scalar product, denoted by $\vec{a}$, $\vec{b}$, is defined by

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

Clearly, $\vec{a}\cdot\vec{b}$ is a scalar as $|\vec{a}|$, $|\vec{b}|$ and $\cos\theta$ are all scalars.

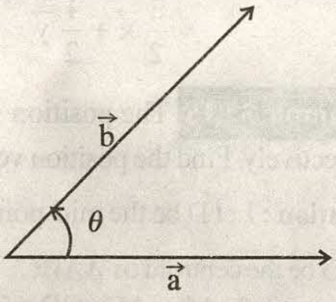


Fig. 32.27

Remarks

1. If $\vec{a}$ and $\vec{b}$ are like vectors, then $\vec{a}\cdot\vec{b} = ab\cos\theta = ab$, where a and b are magnitudes of $\vec{a}$ and $\vec{b}$.

2. If $\vec{a}$ and $\vec{b}$ are unlike vectors, then $\vec{a}\cdot\vec{b} = ab\cos\pi = -ab$

4. Angle θ between the vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

5. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ and $\vec{a}\cdot(\vec{b}+\vec{c}) = (\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c})$.

6. $n(\vec{a}\cdot\vec{b}) = (n\,\vec{a})\cdot\vec{b} = \vec{a}\cdot(n\,\vec{b})$ where n is any real number.

7. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ as $\hat{i}$, $\hat{j}$ and $\hat{k}$ are mutually perpendicular unit vectors.

Example 32.19 If $\vec{a} = 3\hat{i} + 2\hat{j} - 6\hat{k}$ and $\vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$, find $\vec{a}\cdot\vec{b}$. Also find angle between $\vec{a}$ and $\vec{b}$.

Solution : $\vec{a}\cdot\vec{b} = (3\hat{i} + 2\hat{j} - 6\hat{k})\cdot(4\hat{i} - 3\hat{j} + \hat{k})$

$$= 3\times 4 + 2\times(-3) + (-6)\times 1$$

$$\left[\because\ \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \text{ and } \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0\right]$$

$$= 12 - 6 - 6 = 0$$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$

Then $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = 0$

$\therefore\quad \theta = \dfrac{\pi}{2}$.

32.13 VECTOR PRODUCT OF TWO VECTORS

Before we define vector product of two vectors, we discuss below right handed and left handed screw and associate it with corresponding vector triad.

32.13.1 Right Handed Screw

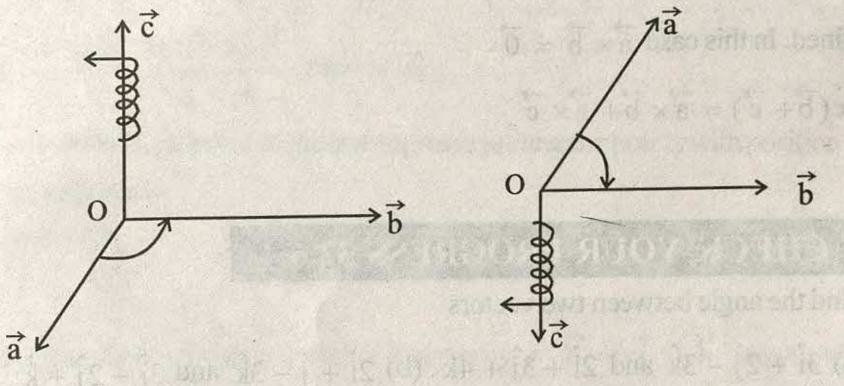
If a screw is taken and rotated in the anticlockwise direction, it translates towards the reader. It is called ***right handed screw***.

32.13.2 Left handed Screw

If a screw is taken and rotated in the clockwise direction, it translates away from the reader. It is called a left handed screw.

Now we associate a screw with given ordered vector triad.

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors whose initial point is O.



(i) (ii)

Fig. 32.28

Now if a right handed screw at O is rotated from $\vec{a}$ towards $\vec{b}$ through an angle <180°, it will undergo a translation along $\vec{c}$ [Fig. 32.28 (i)]

Similarly if a left handed screw at O is rotated from $\vec{a}$ to $\vec{b}$ through an angle <180°, it will undergo a translation along $\vec{c}$ [Fig. 32.28 (ii)]. This time the direction of translation will be opposite to the first one.

Thus an ordered vector triad $\vec{a}, \vec{b}, \vec{c}$ is said to be right handed or left handed according as the right handed screw translated along $\vec{c}$ or opposite to $\vec{c}$ when it is rotated through an angle less than 180°

32.13.3 Vector product

Let $\vec{a}$ and $\vec{b}$ be two vectors and θ be the angle between them such that $0 < \theta < \pi$. The vector product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and is defined as the vector

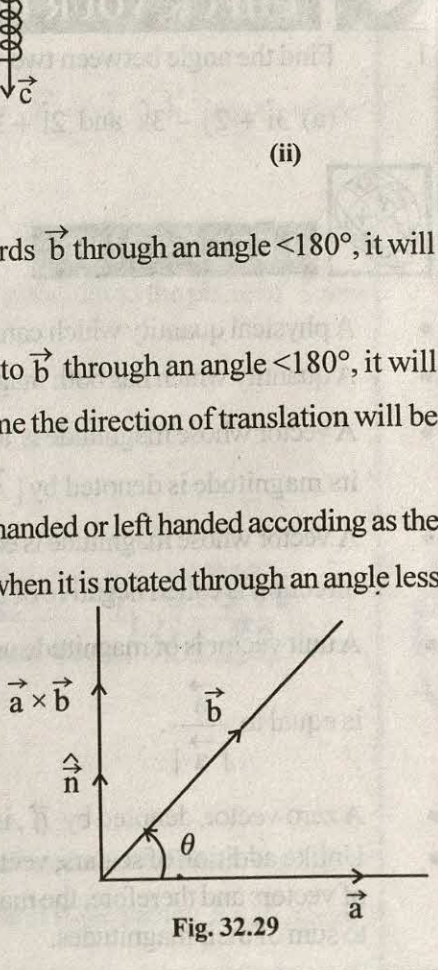


Fig. 32.29

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$ where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed triad of vectors.

Remark :

1. Clearly $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $\vec{a} \times \vec{a} = \vec{0}$
3. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$
4. $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$, and $\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$, $\hat{i} \times \hat{k} = -\hat{j}$
5. If $\vec{a} \times \vec{b} = \vec{0}$, then either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \parallel \vec{b}$.
6. θ is not defined if any or both of $\vec{a}$ and $\vec{b}$ are $\vec{0}$. As $\vec{0}$ has no direction and so $\hat{n}$ is not defined. In this case $\vec{a} \times \vec{b} = \vec{0}$.
7. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

CHECK YOUR PROGRESS 32.6

1. Find the angle between two vectors

   (a) $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $2\hat{i} + 3\hat{j} + 4\hat{k}$. (b) $2\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$

LET US SUM UP

- A physical quantity which can be represented by a number only is called a scalar.
- A quantity which has both magnitude and direction is called a vector.
- A vector whose magnitude is 'a' and direction from A to B can be represented by $\overrightarrow{AB}$ and its magnitude is denoted by $|\overrightarrow{AB}| = a$.
- A vector whose magnitude is equal to the magnitude of another vector $\vec{a}$ but of opposite direction is called negative of the given vector and is denoted by $-\vec{a}$.
- A unit vector is of magnitude unity. Thus, a unit vector parallel to $\vec{a}$ is denoted by $\hat{a}$ and is equal to $\dfrac{\vec{a}}{|\vec{a}|}$.
- A zero vector, denoted by $\vec{0}$, is of magnitude 0 while it has no definite direction.
- Unlike addition of scalars, vectors are added in accordance with triangle law of addition of vectors and therefore, the magnitude of sum of two vectors is always less than or equal to sum of their magnitudes.

- Two or more vectors are said to be collinear if their supports are the same or parallel.
- Three or more vectors are said to be coplanar if their supports are parallel to the same plane or lie on the same plane.
- If $\vec{a}$ is a vector and x is a scalar, then $x\vec{a}$ is a vector whose magnitude is $|x|$ times the magnitude of $\vec{a}$ and whose direction is the same or opposite to that of $\vec{a}$ depending upon $x > 0$ or $x < 0$.
- Any vector co-planar with two given non-collinear vectors is expressible as their linear combination.
- Any vector in space is expressible as a linear combination of three given non-coplanar vectors.
- The position vector of a point that divides the line segment joining the points with position vectors $\vec{a}$ and $\vec{b}$ in the ratio of m : n internally/externally are given by
- $\dfrac{n\vec{a}+m\vec{b}}{m+n}, \dfrac{n\vec{a}-m\vec{b}}{n-m}$ respectively.
- The position vector of mid-point of the line segment joining the points with position vectors $\vec{a}$ and $\vec{b}$ is given by

$$\frac{\vec{a}+\vec{b}}{2}$$

- The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.
- The vector product of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where θ is the angle between $\vec{a}$, $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

SUPPORTIVE WEBSITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

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TERMINAL EXERCISE

1. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors such that any two of them are non-collinear. Find their sum if the vector $\vec{a}+\vec{b}$ is collinear with the vector $\vec{c}$ and if the vector $\vec{b}+\vec{c}$ is collinear with $\vec{a}$.

2. Prove that any two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exist numbers x and y, both not zero simultaneously, such that $x\,\vec{a} + y\,\vec{b} = \vec{0}$.

3. ABCD is a parallelogram in which M is the mid-point of side CD. Express the vectors $\overrightarrow{BD}$ and $\overrightarrow{AM}$ in terms of vectors $\overrightarrow{BM}$ and $\overrightarrow{MC}$.

4. Can the length of the vector $\vec{a} - \vec{b}$ be (i) less than, (ii) equal to or (iii) larger than the sum of the lengths of vectors $\vec{a}$ and $\vec{b}$?

5. Let $\vec{a}$ and $\vec{b}$ be two non-collinear vectors. Find the number x and y, if the vector $(2 - x)\,\vec{a} + \vec{b}$ and $y\,\vec{a} + (x - 3)\,\vec{b}$ are equal.

6. The vectors $\vec{a}$ and $\vec{b}$ are non-collinear. Find the number x if the vector $3\,\vec{a} + x\,\vec{b}$ and $(1 - x)\,\vec{a} - \frac{2}{3}\,\vec{b}$ are parallel.

7. Determine x and y such that the vector $\vec{a} = -2\hat{i} + 3\hat{j} + y\hat{k}$ is collinear with the vector $\vec{b} = x\hat{i} - 6\hat{j} + 2\hat{k}$. Find also the magnitudes of $\vec{a}$ and $\vec{b}$.

8. Determine the magnitudes of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ if $\vec{a} = 3\hat{i} - 5\hat{j} + 8\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - 4\hat{k}$.

9. Find a unit vector in the direction of $\vec{a}$ where $\vec{a} = -6\hat{i} + 3\hat{j} - 2\hat{k}$.

10. Find a unit vector parallel to the resultant of vectors $3\hat{i} - 2\hat{j} + \hat{k}$ and $-2\hat{i} + 4\hat{j} + \hat{k}$

11. The following forces act on a particle P :

    $\vec{F_1} = 2\hat{i} + \hat{j} - 3\hat{k}$, $\vec{F_2} = -3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{F_3} = 3\hat{i} - 2\hat{j} + \hat{k}$ measured in Newtons.

    Find (a) the resultant of the forces, (b) the magnitude of the resultant.

12. Show that the following vectors are co-planar :

    $(\vec{a} - 2\,\vec{b} + \vec{c})$, $(2\,\vec{a} + \vec{b} - 3\,\vec{c})$ and $(-3\,\vec{a} + \vec{b} + 2\,\vec{c})$

    where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three non-coplanar vectors.

ANSWERS

CHECK YOUR PROGRESS 32.1

1. (d) 2. (b)

3.

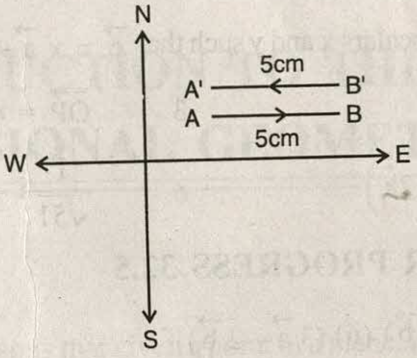

Fig. 32.30

4. Two vectors are said to be like if they have same direction what ever be their magnitudes. But in case of equal vectors magnitudes and directions both must be same.

5.

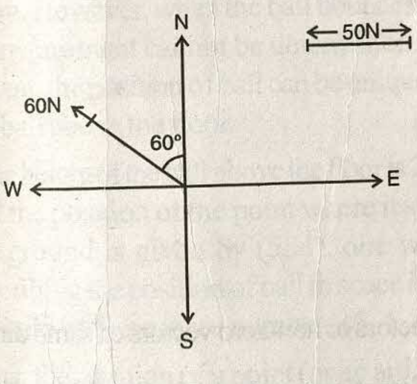

Fig. 32.31

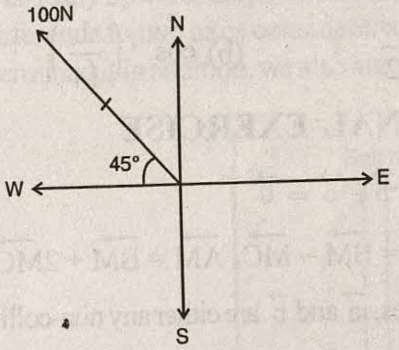

Fig. 32.32

CHECK YOUR PROGRESS 32.2

1. $\vec{0}$ 2. $\vec{0}$

CHECK YOUR PROGRESS 32.3

1. $\vec{b} - \vec{a}$

2. (i) It is a vector in the direction of $\vec{a}$ and whose magnitudes is 3 times that of $\vec{a}$.

   (ii) It is a vector in the direction opposite to that of $\vec{b}$ and with magnitude 5 times that of $\vec{b}$.

3. $\overrightarrow{DB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = 2\vec{a} + 3\vec{b}$.

4. $|y\vec{n}| = y|\vec{n}|$ if $y > 0$
   $= -y|\vec{n}|$ if $y < 0$
   $= 0$ if $y = 0$

5. Vector

6. $\vec{p} = x\vec{q}$, x is a non-zero scalar.

CHECK YOUR PROGRESS 32.4

1. If there exist scalars x and y such that $\vec{c} = x\vec{a} + y\vec{b}$

2. $\vec{r} = 3\hat{i} + 4\hat{j}$

3. $\overrightarrow{OP} = 4\hat{i} + 3\hat{j} + 5\hat{k}$

5. $\frac{1}{7}\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right)$

6. $\frac{1}{\sqrt{51}}\hat{i} - \frac{5}{\sqrt{51}}\hat{j} - \frac{5}{\sqrt{51}}\hat{k}$

CHECK YOUR PROGRESS 32.5

1. (i) $\frac{1}{5}(2\vec{a} + 3\vec{b})$ (ii) $(3\vec{a} - 2\vec{b})$

2. $\frac{1}{7}(4\vec{p} + 3\vec{q})$

3. $\frac{1}{3}(2\vec{c} + \vec{d}), \frac{1}{3}(\vec{c} + 2\vec{d})$

CHECK YOUR PROGRESS 32.6

1. (a) $\frac{\pi}{2}$ (b) $\cos^{-1}\left(\frac{1}{14}\right)$

TERMINAL EXERCISE

1. $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

3. $\overrightarrow{BD} = \overrightarrow{BM} - \overrightarrow{MC}, \overrightarrow{AM} = \overrightarrow{BM} + 2\overrightarrow{MC}$

4. (i) Yes, $\vec{a}$ and $\vec{b}$ are either any non-collinear vectors or non-zero vectors of same direction.

   (ii) Yes, $\vec{a}$ and $\vec{b}$ are either in the opposite directions or at least one of them is a zero vector.

   (iii) Yes, $\vec{a}$ and $\vec{b}$ have opposite directions.

5. $x = 4, \ y = -2$

6. $x = 2, \ -1$

7. $x = 4, \ y = -1$ $\quad |\vec{a}| = \sqrt{14}, |\vec{b}| = 2\sqrt{14}$

8. $|\vec{a} + \vec{b}| = 6, |\vec{a} - \vec{b}| = 14$

9. $-\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{2}{7}\hat{k}$

10. $\pm\frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

11. $2\hat{i} + \hat{j}; \sqrt{5}$

33

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

You have read in your earlier lessons that given a point in a plane, it is possible to find two numbers, called its co-ordinates in the plane. Conversely, given any ordered pair (x, y) there corresponds a point in the plane whose co-ordinates are (x, y).

Let a rubber ball be dropped vertically in a room The point on the floor, where the ball strikes, can be uniquely determined with reference to axes, taken along the length and breadth of the room. However, when the ball bounces back vertically upward, the position of the ball in space at any moment cannot be determined with reference to two axes considered earlier. At any instant, the position of ball can be uniquely determined if in addition, we also know the height of the ball above the floor.

If the height of the ball above the floor is 2.5cm and the position of the point where it strikes the ground is given by (5, 4), one way of describing the position of ball in space is with the help of these three numbers (5, 4, 2.5).

Thus, the position of a point (or an article) in space can be uniquely determined with the help of three numbers. In this lesson, we will discuss in details about the co-ordinate system and co-ordinates of a point in space, distance between two points in space, position of a point dividing the join of two points in a given ratio internally/externally and about the projection of a point/line in space.

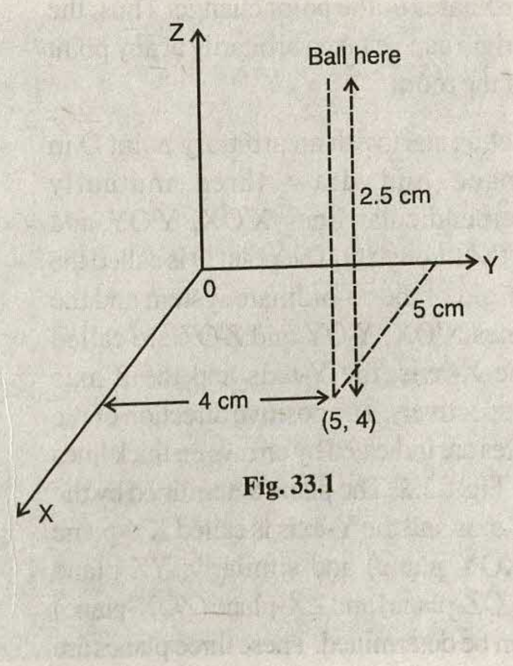

Fig. 33.1

OBJECTIVES

After studying this lesson, you will be able to :

- associate a point, in three dimensional space with given triplet and vice versa;
- find the distance between two points in space;

OPTIONAL - I
Vectors and three
imensional Geometry

Notes

- find the coordinates of a point which divides the line segment joining twogiven points in a given ratio internally and externally;
- define the direction cosines/ratios of a given line in space;
- find the direction cosines of a line in space;
- find the projection of a line segment on another line; and
- find the condition of prependicularity and parallelism of two lines in space.

EXPECTED BACKGROUND KNOWLEDGE

- Two dimensional co-ordinate geometry
- Fundamentals of Algebra, Geometry, Trigonometry and vector algebra.

33.1 COORDINATE SYSTEM AND COORDINATES OF A POINT IN SPACE

Recall the example of a bouncing ball in a room where one corner of the room was considered as the origin.

It is not necessary to take a particular corner of the room as the origin. We could have taken any corner of the room (for the matter any point of the room) as origin of reference, and relative to that the co-ordinates of the point change. Thus, the origin can be taken arbitarily at any point of the room.

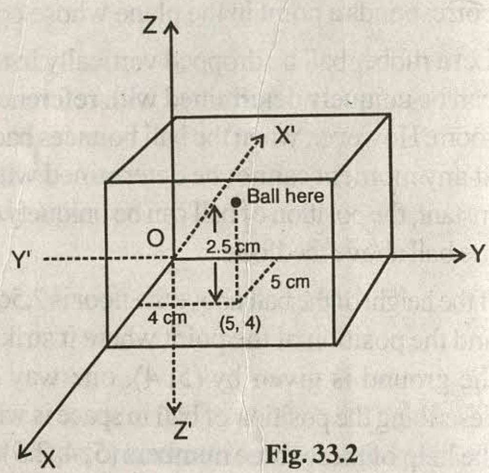

Fig. 33.2

Let us start with an arbitrary point O in space and draw three mutually perpendicular lines X'OX, Y'OY and Z'OZ through O. The point O is called the origin of the co-ordinate system and the lines X'OX, Y'OY and Z'OZ are called the X-axis, the Y-axis and the Z-axis respectively. The positive direction of the axes are indicated by arrows on thick lines in Fig. 33.2. The plane determined by the X-axis and the Y-axis is called XY-plane (XOY plane) and similarly, YZ-plane (YOZ-plane) and ZX-plane (ZOX-plane) can be determined. These three planes are called co-ordinate planes. The three coordinate planes divide the whole space into eight parts called **octants**.

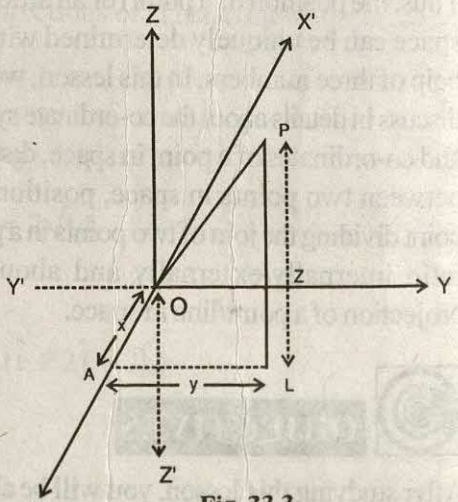

Fig. 33.3

Let P be any point is space. Through P draw perpendicular PL on XY-plane meeting this plane at L. Through L draw a line LA parallel to OY cutting OX in A. If we write OZ = x, AL = y and LP = z, then (x, y, z) are the co-ordinates of the point P.

Again, if we complete a reactangular parallelopiped through P with its three edges OA, OB and OC meeting each other at O and OP as its main diagonal then the lengths (OA, OB, OC) i.e., (x, y, z) are called the co-ordinates of the point P.

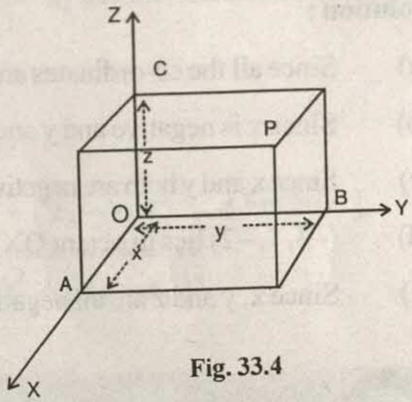

Fig. 33.4

Note : You may note that in Fig. 33.4

(i) The x co-ordinate of P = OA = the length of perpendicular from P on the YZ-plane.

(ii) The y co-ordinate of P = OB = the length of perpendicular from P on the ZX-plane.

(iii) The x co-ordinate of P = OC = the length of perpendicular from P on the XY-plane.

Thus, the co-ordinates x, y, and z of any point are the perpendicular distances of P from the three rectangular co-ordinate planes YZ, ZX and XY respectively.

Thus, given a point P in space, to it corresponds a triplet (x, y, z) called the co-ordinates of the point in space. Conversely, given any triplet (x, y, z), there corresponds a point P in space whose co-ordinates are (x, y, z).

Remarks

1. Just as in plane co-ordinate geometry, the co-ordinate axes divide the plane into four quadrants, in three dimentional geometry, the space is divided into eight octants by the co-ordinate planes, namely OXYZ, OX'YZ, OXY'Z, OXYZ', OXY'Z', OX'YZ', OX'Y'Z and OX'Y'Z'.
2. If P be any point in the first octant, there is a point in each of the other octants whose absolute distances from the co-ordinate planes are equal to those of P. If P be (a, b, c), the other points are (–a, b, c), (a, –b, c), (a, b, –c), (a, –b, –c), (–a, b, –c), (–a, –b, c) and (–a, –b, –c) respectively in order in the octants referred in (i).
3. The co-ordinates of point in XY-plane, YZ-plane and ZX-plane are of the form (a, b, 0), (0, b, c) and (a, 0, c) respectively.
4. The co-ordinates of points on X-axis, Y-axis and Z-axis are of the form (a, 0, 0), (0, b, 0) and (0, 0, c) respectively.
5. You may see that (x, y, z) corresponds to the position vector of the point P with reference to the origin O as the vector $\overrightarrow{OP}$.

Example 33.1 Name the octant wherein the given points lies :

(a) (2, 6, 8)   (b) (–1, 2, 3)   (c) (–2, –5, 1)

(d) (–3, 1, –2)   (e) (–6, –1, –2)

Solution :

(a) Since all the co-ordinates are positive, $\therefore$ (2, 6, 8) lies in the octant OXYZ.

(b) Since x is negative and y and z are positive, $\therefore$ (–1, 2, 3) lies in the octant OX'YZ.

(c) Since x and y both are negative and z is positive $\therefore$ (–2, –5, 1) lies in the octant OX'Y'Z.

(d) (–3, 1, –2) lies in octant OX'YZ'.

(e) Since x, y and z are all negative $\therefore$ (–6, –1, –2) lies in the octant OX'Y'Z'.

CHECK YOUR PROGRESS 33.1

1. Name the octant wherein the given points lies :

(a) (–4, 2, 5) (b) (4, 3, –6) (c) (–2, 1, –3)

(d) (1, –1, 1) (f) (8, 9, –10)

33.2 DISTANCE BETWEEN TWO POINTS

Suppose there is an electric plug on a wall of a room and an electric iron placed on the top of a table. What is the shortest length of the wire needed to connect the electric iron to the electric plug ? This is an example necessitating the finding of the distance between two points in space.

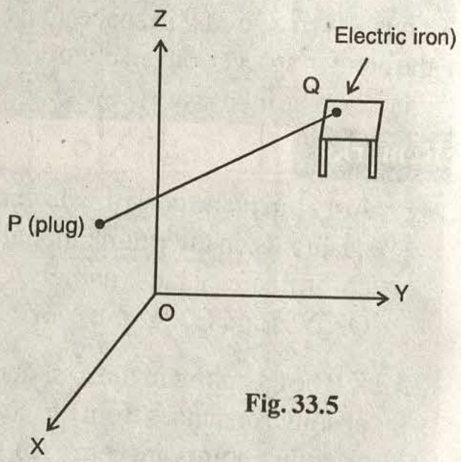

Fig. 33.5

Let the co-ordinates of two points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. With PQ as diagonal, complete the parallopiped PMSNRLKQ.

PK is perpendicular to the line KQ.

$\therefore$ From the right-angled triangle PKQ, right angled at K,

We have $PQ^2 = PK^2 + KQ^2$

Again from the right angled triangle PKL right angled at L,

$$PK^2 = KL^2 + PL^2 = MP^2 + PL^2 \quad (\because KL = MP)$$

$$\therefore \quad PQ^2 = MP^2 + PL^2 + KQ^2 \quad ...(i)$$

The edges MP, PL and KQ are parallel to the co-ordinate axes.

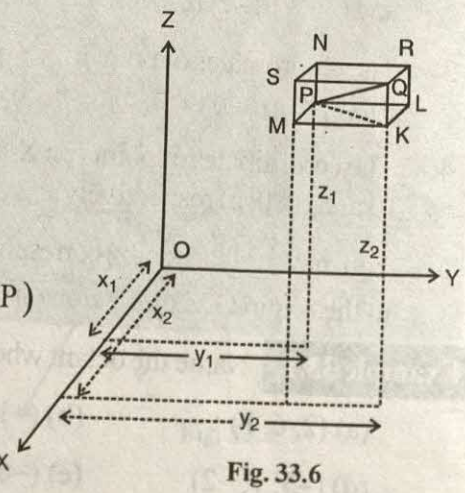

Fig. 33.6

Now, the distance of the point P from the plane $YOZ = x_1$ and the distance of Q and M from YOZ plane $= x_2$

$\therefore$ $$MP = |x_2 - x_1|$$

Similarly, $$PL = |y_2 - y_1| \text{ and } KQ = |z_2 - z_1|$$

$\therefore$ $$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad \text{....[From (i)]}$$

or $$|\mathbf{PQ}| = \sqrt{(\mathbf{x_2} - \mathbf{x_1})^2 + (\mathbf{y_2} - \mathbf{y_1})^2 + (\mathbf{z_2} - \mathbf{z_1})^2}$$

Corollary : Distance of a Point from the Origin

If the point $Q(x_2, y_2, z_2)$ coincides with the origin (0, 0, 0), then $x_2 = y_2 = z_2 = 0$

$\therefore$ The distance of P from the origin is

$$|OP| = \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2 + (z_1 - 0)^2}$$

$$= \sqrt{x_1^2 + y_1^2 + z_1^2}$$

In general, the distance of a point P(x, y, z) from origin O is given by

$$|OP| = \sqrt{x^2 + y^2 + z^2}$$

Example 33.2 Find the distance between the points $(2, 5, -4)$ and $(8, 2, -6)$.

Solution : Let P $(2, 5, -4)$ and Q $(8, 2, -6)$ be the two given points.

$$\therefore \quad |PQ| = \sqrt{(8-2)^2 + (2-5)^2 + (-6+4)^2}$$

$$= \sqrt{36 + 9 + 4}$$

$$= \sqrt{49}$$

$$= 7$$

$\therefore$ The distance between the given points is 7 units.

Example 33.3 Prove that the points $(-2, 4, -3)$, $(4, -3, -2)$ and $(-3, -2, 4)$ are the vertices of an equilateral triangle.

Solution : Let $A(-2, 4, -3)$, $B(4, -3, -2)$ and $C(-3, -2, 4)$ be the three given points.

Now $$|AB| = \sqrt{(4+2)^2 + (-3-4)^2 + (-2+3)^2}$$

$$= \sqrt{36 + 49 + 1} = \sqrt{86}$$

$$|BC| = \sqrt{(-3-4)^2 + (-2+3)^2 + (4+2)^2} = \sqrt{86}$$

$$|CA| = \sqrt{(-2+3)^2 + (4+2)^2 + (-3-4)^2} = \sqrt{86}$$

Since $|AB| = |BC| = |CA|$, Δ ABC is an equilateral triangle.

Example 33.4 Verify whether the following points form a triangle or not :

(a) $A(-1, 2, 3)$ $B(1, 4, 5)$ and $C(5, 4, 0)$

(b) $(2, -3, 3), (1, 2, 4)$ and $(3, -8, 2)$

Solution :

$$\text{(a) } |AB| = \sqrt{(1+1)^2 + (4-2)^2 + (5-3)^2}$$

$$= \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3} \quad = 3.464 \text{ (approx.)}$$

$$|BC| = \sqrt{(5-1)^2 + (4-4)^2 + (0-5)^2}$$

$$= \sqrt{16 + 0 + 25} = \sqrt{41} \quad = 6.4 \text{ (approx.)}$$

and $$|AC| = \sqrt{(5+1)^2 + (4-2)^2 + (0-3)^2}$$

$$= \sqrt{36 + 4 + 9} = 7$$

$\therefore$ $|AB| + |BC| = 3.464 + 6.4 = 9.864 > |AC|$, $|AB| + |AC| > |BC|$

and $|BC| + |AC| > |AB|$.

Since sum of any two sides is greater than the third side, therefore the above points form a triangle.

(b) Let the points $(2, -3, 3), (1, 2, 4)$ and $(3, -8, 2)$ be denoted by P, Q and R respectively,

then $$|PQ| = \sqrt{(1-2)^2 + (2+3)^2 + (4-3)^2}$$

$$= \sqrt{1 + 25 + 1} = 3\sqrt{3}$$

$$|QR| = \sqrt{(3-1)^2 + (-8-2)^2 + (2-4)^2}$$

$$= \sqrt{4 + 100 + 4} = 6\sqrt{3}$$

$$|PR| = \sqrt{(3-2)^2 + (-8+3)^2 + (2-3)^2}$$

$$= \sqrt{1 + 25 + 1}$$

$$= 3\sqrt{3}$$

In this case $|PQ| + |PR| = 3\sqrt{3} + 3\sqrt{3} = 6\sqrt{3} = |QR|$. Hence the given points do not form a triangle. In fact the points lie on a line.

Example 33.5 Show that the points $A(1,2,-2), B(2,3,-4)$ and $C(3,4,-3)$ form a right angled triangle.

Solution : $AB^2 = (2-1)^2 + (3-2)^2 + (-4+2)^2 = 1+1+4 = 6$

$$BC^2 = (3-2)^2 + (4-3)^2 + (-3+4)^2 = 1+1+1 = 3$$

and $AC^2 = (3-1)^2 + (4-2)^2 + (-3+2)^2 = 4+4+1 = 9$

We observe that $AB^2 + BC^2 = 6 + 3 = 9 = AC^2$

$\therefore$ Δ ABC is a right angled triangle.

Hence the given points form a right angled triangle.

Example 33.6 Prove that the points $A(0, 4, 1)$, $B(2, 3, -1)$, $C(4, 5, 0)$ and $D(2, 6. 2)$ are vertices of a square.

Solution : Here, $AB = \sqrt{(2-0)^2 + (3-4)^2 + (-1-1)^2}$

$$= \sqrt{4+1+4} = 3 \text{ units}$$

$$BC = \sqrt{(4-2)^2 + (5-3)^2 + (0+1)^2}$$

$$= \sqrt{4+4+1} = 3 \text{ units}$$

$$CD = \sqrt{(2-4)^2 + (6-5)^2 + (2-0)^2}$$

$$= \sqrt{4+1+4} = 3 \text{ units}$$

and $DA = \sqrt{(0-2)^2 + (4-6)^2 + (1-2)^2}$

$$= \sqrt{4+4+1} = 3 \text{ units}$$

$$\therefore \quad AB = BC = CD = DA$$

Now $AC^2 = \sqrt{(4-0)^2 + (5-4)^2 + (0-1)^2}$

$$= 16 + 1 + 1 = 18$$

$$\therefore \quad AB^2 + BC^2 = 3^2 + 3^2 = 18 = AC^2$$

$$\therefore \quad \angle B = 90°$$

$\therefore$ In quadrilateral A B C D, $AB = BC = CD = DA$ and $\angle B = 90°$

$\therefore$ A B C D is a square.

CHECK YOUR PROGRESS 33.2

1. Find the distance between the following points :

(a) $(4, 3, -6)$ and $(-2, 1, -3)$ (b) $(-3, 1, -2)$ and $(-3, -1, 2)$

Notes

(c) $(0, 0, 0)$ and $(-1, 1, 1)$

2. Show that if the distance between the points $(5, -1, 7)$ and $(a, 5, 1)$ is 9 units, "a" must be either 2 or 8.

3. Show that the triangle formed by the points (a, b, c), (b, c, a) and (c, a, b) is equilateral.

4. Show that the the points $(-1, 0, -4)$, $(0, 1, -6)$ and $(1, 2, -5)$ form a right angled tringle.

5. Show that the points $(0, 7, 10)$, $(-1, 6, 6)$ and $(-4, 9, 6)$ are the vertices of an isosceles right-angled triangle.

6. Show that the points $(3, -1, 2)$, $(5, -2, -3)$, $(-2, 4, 1)$ and $(-4, 5, 6)$ form a parallelogram.

7. Show that the points $(2, 2, 2)$, $(-4, 8, 2)$, $(-2, 10, 10)$ and $(4, 4, 10)$ form a square.

8. Show that in each of the following cases the three points are collinear :

   (a) $(-3, 2, 4)$, $(-1, 5, 9)$ and $(1, 8, 14)$

   (b) $(5, 4, 2)$, $(6, 2, -1)$ and $(8, -2, -7)$

   (c) $(2, 5, -4)$, $(1, 4, -3)$ and $(4, 7, -6)$

33.3 COORDINATES OF A POINT OF DIVISION OF A LINE SEGMENT

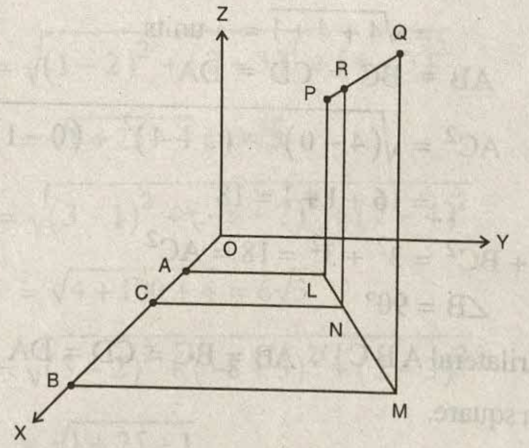

Fig. 33.7

Let the point R (x, y, z) divide PQ in the ratio *l* : *m* internally.
Let the co-ordinates of P be (x_1, y_1, z_1) and the co-ordinates of Q be (x_2, y_2, z_2). From points P, R and Q, draw PL, RN and QM perpendiculars to the XY-plane.

Notes

Draw LA, NC and MB perpendiculars to OX.

Now $AC = OC - OA = x - x_1$ and $BC = OB - OC = x_2 - x$

Also we have, $\dfrac{AC}{BC} = \dfrac{LN}{NM} = \dfrac{PR}{RQ} = \dfrac{l}{m}$

$$\therefore \qquad \frac{x - x_1}{x_2 - x} = \frac{l}{m}$$

or $$mx - mx_1 = lx_2 - lx$$

or $$(l + m)x = lx_2 + mx_1$$

or $$x = \frac{lx_2 - mx_1}{l + m}$$

Similarly, if we draw perpendiculars to OY and OZ respectively,

we get $y = \dfrac{ly_2 + my_1}{l + m}$ and $z = \dfrac{lz_2 + mz_1}{l + m}$

$$\therefore \quad \text{R is the point} \left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}, \frac{lz_2 + mz_1}{l + m} \right)$$

If $\lambda = \dfrac{l}{m}$, then the co-ordinates of the point R which divides PQ in the ratio $\lambda : 1$ are

$$\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1} \right), \lambda + 1 \neq 0$$

It is clear that to every value of λ, there corresponds a point of the line PQ and to every point R on the line PQ, there corresponds some value of λ. If λ is postive, R lies on the line segment PQ and if λ is negative, R does not lie on line segment PQ.

In the second case you may say the R divides the line segment PQ externally in the ratio $-\lambda : 1$.

Corollary 1 : The co-ordinates of the point dividing PQ externally in the ratio $l : m$ are

$$\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}, \frac{lz_2 - mz_1}{l - m} \right)$$

Corollary 2 : The co-ordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Example 33.7 Find the co-ordinates of the point which divides the line segment joining the points $(2, -4, 3)$ and $(-4, 5, -6)$ in the ratio 2 : 1 internally.

Solution : Let $A(2, -4, 3)$, $B(-4, 5, -6)$ be the two points.

Let P (x, y, z) divides AB in the ratio 2 : 1.

$$\therefore \qquad x = \frac{2(-4)+1.2}{2+1} = -2, \quad y = \frac{2.5+1(-4)}{2+1} = 2$$

and $$z = \frac{2(-6)+1.3}{2+1} = -3$$

Thus, the co-ordinates of P are $(-2, 2, -3)$

Example 33.8 Find the point which divides the line segment joining the points $(-1, -3, 2)$ and $(1, -1, 2)$ externally in the ratio 2 : 3.

Solution : Let the points $(-1, -3, 2)$ and $(1, -1, 2)$ be denoted by P and Q respectively.

Let R (x, y, z) divide PQ externally in the ratio 2 : 3. Then

$$x = \frac{2(1)-3(-1)}{2-3} = -5, \qquad y = \frac{2(-1)-3(-3)}{2-3} = -7$$

and $$z = \frac{2(2)-3(2)}{2-3} = 2$$

Thus, the co-ordinates of R are $(-5, -7, 2)$.

Example 33.9 Find the ratio in which the line segment joining the points $(2-3, 5)$ and $(7, 1, 3)$ is divided by the XY-plane.

Solution : Let the required ratio in which the line segment is divided be $l : m$.

The co-ordinates of the point are $\left(\frac{7l+2m}{l+m}, \frac{l-3m}{l+m}, \frac{3l+5m}{l+m}\right)$

Since the point lies in the XY-plane, its z-coordinate is zero.

i.e., $$\frac{3l+5m}{l+m} = 0 \text{ or } \frac{l}{m} = -\frac{5}{3}$$

Hence the XY-plane divides the join of given points in the ratio 5 : 3 externally.

CHECK YOUR PROGRESS 33.3

1. Find the co-ordinates of the point which divides the line segment joining two points $(2, -5, 3)$ and $(-3, 5, -2)$ internally in the ratio 1 : 4.
2. Find the coordinates of points which divide the join of the points $(2, -3, 1)$ and $(3, 4, -5)$ internally and externally in the ratio 3 : 2.
3. Find the ratio in which the line segment joining the points $(2, 4, 5)$ and $(3, 5, -4)$ is divided by the YZ-plane.

4. Show that the YZ-plane divides the line segment joining the points $(3, 5, -7)$ and $(-2, 1, 8)$ in the ration 3 : 2 at the point $\left(0, \frac{13}{5}, 2\right)$

5. Show that the ratios in which the co-ordinate planes divide the join of the points $(-2, 4, 7)$ and $(3, -5, 8)$ are 2 : 3, 4 : 5 (internally) and 7 : 8 (externally).

6. Find the co-ordinates of a point R which divides the line segment $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio 2 : 1. Verify that Q is the mid-point of PR.

33.4 ANGLE BETWEEN TWO LINES

You are already famililar with the concept of the angle between two lines in plane geometry. We will extend this idea to the lines in space.

Let there be two lines in space, intersecting or non-intersecting. We consider a point A in space and through it, we draw lines parallel to the given lines in space. The angle between these two lines drawn parallel to the given lines is defined as the angle between the two lines in space.

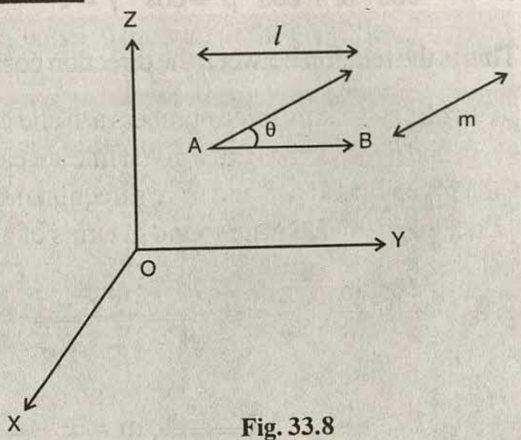

Fig. 33.8

You may see in the adjointing figure, that θ is the angle between the lines l and m.

33.5 DIRECTION COSINES OF A LINE

If α, β and γ are the angles which a line AB makes with the positive directions of X, Y and Z axes respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the direction cosines of the line AB and are usually denoted by the letters l, m and n respectively. In other words $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$. You may easily see that the direction cosines of the X-axis are 1, 0, 0, because the line coincides with the X axis and is perpendicular to Y and Z axes since $\cos 0° = 1$, $\cos 90° = 0$. Similarly direction cosines of Y and Z axes are 0, 1, 0 and 0, 0, 1 respectively.

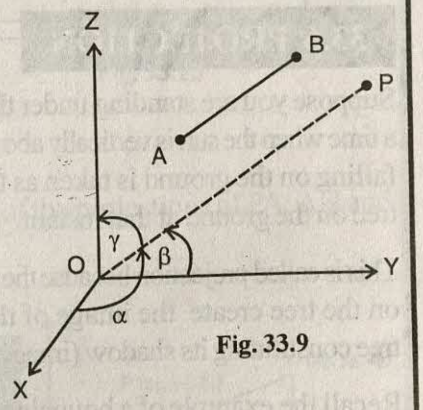

Fig. 33.9

33.5.1 RELATION BETWEEN DRECTION COSINES

Let OP be a line with direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ i.e. l, m and n.

Again since each of $\angle OLP$, $\angle OMP$ and $\angle ONP$ is a right angle.

We have, $\frac{OL}{OP} = \cos\alpha = l$

$$\frac{OM}{OP} = \cos\beta = m \text{ and } \frac{ON}{OP} = \cos\gamma = n$$

i.e. $l.OP = OL$, $m.OP = OM$ and $n.OP = ON$

$$\therefore \quad OP^2\left(l^2 + m^2 + n^2\right)$$

$$= OL^2 + OM^2 + ON^2 = OP^2$$

or $$l^2 + m^2 + n^2 = 1$$

or $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

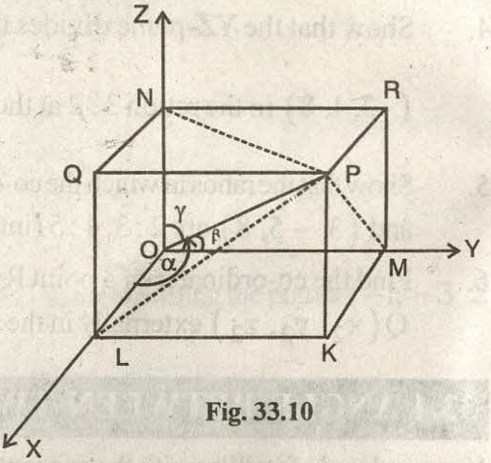

Fig. 33.10

This is the relation between the direction cosines of a line.

Corollary 1 : Any three numbers a, b and c which are proportional to the direction cosines l, m and n respectively of a given line are called the direction ratios or direction numbers of the given line. If a, b and c are direction numbers and l, m and n are direction cosines of a line, then l, m and n are found in terms of a, b and c as follows :

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{l^2 + m^2 + n^2}{\sqrt{a^2 + b^2 + c^2}} = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}} =$$

$$\therefore \quad l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where the same sign either positive or negative is to be taken throughout.

33.6 PROJECTION

Suppose you are standing under the shade of a tree. At a time when the sun is vertically above the tree, its shadow falling on the ground is taken as the projection of the tree on the ground at that instant.

This is called projection because the rays falling vertically on the tree create the image of the each point of the tree constituting its shadow (image).

Recall the example of a bouncing ball. When the ball falling freely from a point in space strikes the ground, the point where the ball strikes the ground is the projection of the point in space on the ground.

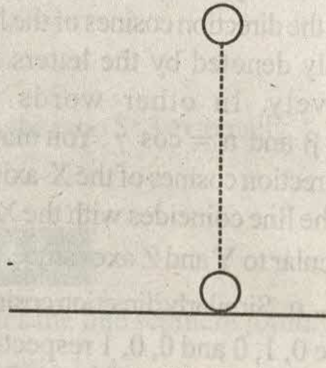

Fig. 33.11

33.6.1 Projection of a Point and of a Line Segment

The projection of a point on a plane can be taken as the foot of the perpendicular drawn from the point to the plane. Similarly, the line segment obtained by joining the feet of the perpendiculars in the plane drawn from the end points of a line segment is called the projection of the line segment on the plane.

We may similarly define the projection of a point and of a line segment on a given line.

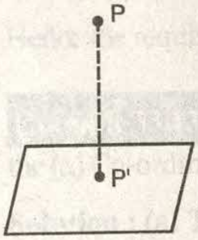

Fig. 33.12

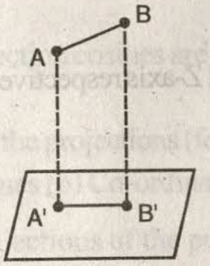

Fig. 33.13

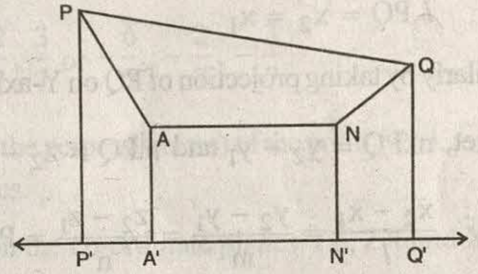

Fig. 33.14

Note : Projection of a line segment PQ on a line is equal to the sum of the projections of the broken line segments i.e., Projections of PQ = Sum of the projections of PA, AN and NQ.

33.6.2 Projection of a Line Segment Joning Two Points on a Line

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. To find the projection of PQ on a line with direction cosines *l*, m and n, through P and Q draw planes parallel to the co-ordinates planes to form a reactangular paralleloppied whose diagonal is PQ.

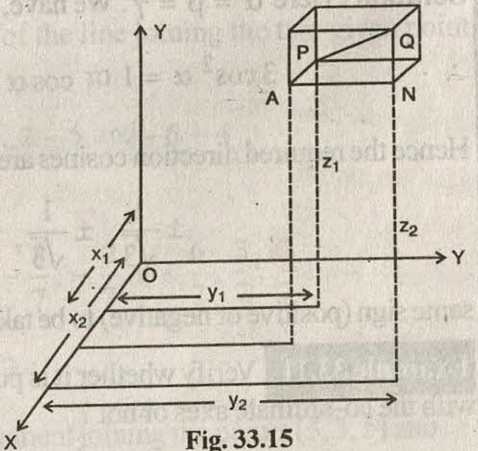

Fig. 33.15

Now $PA = x_2 - x_1$, $AN = y_2 - y_1$

and $NQ = z_2 - z_1$.

The lines PA, AN and NQ are parallel to X-axis, Y-axis and Z-axis respectively.

Therefore, their respective projections on the line with direction cosines *l*, m and n are $(x_2 - x_1)\,l$, $(y_2 - y_1)\,m$ and $(z_2 - z_1)\,n$.

Recall that projection of PQ on any line is equal to the sum of the projections of PA, AN and NQ on the line, therefore the required projection is

$$(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n.$$

33.7 DIRECTION COSINES OF THE LINE JOINING TWO POINTS

Let L and M be the feet of the perpendiculars drawn from $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on the X-axis respectively, so that $OL = x_1$ and $OM = x_2$.

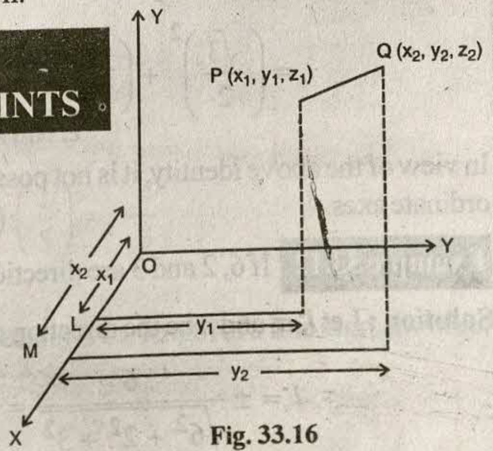

Fig. 33.16

Projection of PQ on X-axis

$$= LM = OM - OL$$

$$= x_2 - x_1$$

Also, if l, m and n are the direction cosines of PQ, the projection of PQ on X-axis = l. PQ

$$\therefore \quad l\ PQ = x_2 = x_1$$

Similarly by taking projection of PQ on Y-axis and Z-axis respectively,

we get, $m.PQ = y_2 - y_1$ and $n.PQ = z_2 - z_1$

$$\therefore \quad \frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = PQ$$

Thus, the direction cosines of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$.

Example 33.10 Find the direction cosines of a line that makes equal angles with the axes.

Solution : Here $\alpha = \beta = \gamma$. We have, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\therefore \quad 3\cos^2\alpha = 1 \text{ or } \cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Hence the required direction cosines are

$$\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$$

same sign (positive or negative) to be taken throughout.

Example 33.11 Verify whether it is possible for a line to make the angles 30°, 45° and 60° with the co-ordinate axes or not ?

Soluton : Let the line make angles α, β and γ with the co-ordinate axes.

$$\alpha = 30°, \beta = 45° \text{ and } \gamma = 60°$$

Since the relation between direction cosines is $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we have

$$\cos^2 30° + \cos^2 45° + \cos^2 60°$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{6}{4} > 1$$

In view of the above identity, it is not possible for a line to make the given angles with the co-ordinate axes.

Example 33.12 If 6, 2 and 3 are direction ratios of a line, find its direction cosines.

Solution : Let l, m and n be the direction cosines of the line.

$$\therefore \quad l = \pm\frac{6}{\sqrt{6^2 + 2^2 + 3^2}} = \pm\frac{6}{7}, \quad m = \pm\frac{2}{\sqrt{6^2 + 2^2 + 3^2}} = \pm\frac{2}{7}$$

and $$n = \pm\frac{3}{\sqrt{6^2+2^2+3^2}} = \pm\frac{3}{7}$$

Hence the required direction cosines are $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$ or $-\frac{6}{7}, -\frac{2}{7}, -\frac{3}{7}$.

Notes

Example 33.13 Find the projections (feet of the perpendiculars) of the point $(2, 1, -3)$ on the (a) Co-ordinate planes (b) Co-ordinate axes.

Solution : (a) The projections of the point on the co-ordinate planes YZ, ZX and XY are $(0, 1, -3)$, $(2, 0, -3)$ and $(2, 1, 0)$ respectively.

(b) The projections on the co-ordinate axes are $(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, -3)$ respectively.

Example 33.14 Find the direction cosines of the line-segment joining the points $(2, 5, -4)$ and $(8, 2, -6)$.

Solution : Let l, m and n be the direction cosines of the line joining the two given points $(2, 5, -4)$ and $(8, 2, -6)$.

Then the direction cosines are proportional to $8-2$, $2-5$ and $-6+4$

i.e., $6, -3, -2$ are direction ratios of the line.

$\therefore$ The required direction cosines of the line are $\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}$ or $-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}$

Since $$\sqrt{(6)^2+(-3)^2(-2)^2} = 7.$$

Example 33.15 Find the projection of the line segment joining the points (3, 3, 5) and (5, 4, 3) on the line joining the points $(2, -1, 4)$ and (0, 1, 5).

Solution : The direction cosines of the line joining the points $(2, -1, 4)$ and (0, 1, 5) are

$$-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$$

because $$\sqrt{(2-0)^2+(-1-1)^2+(4-5)^2} = 3$$

Thus, the projection of this line segment on the given line is

$$(5-3)\left(-\frac{2}{3}\right)+(4-3)\left(\frac{2}{3}\right)+(3-5)\left(\frac{1}{3}\right) = -\frac{4}{3}$$

Hence the required projection is $\frac{4}{3}$ because the projection is the length of a line segment which is always taken as positive.

CHECK YOUR PROGRESS 33.4

1. Find the direction cosines of the line having direction ratios

   (a) 3, –1, 2 (b) 1, 1, 1

2. Find the projections of the point (–3, 5, 6) on the

   (a) Co-ordinate palnes (b) Co-ordinate axes

3. Find the direction cosines of the line segment joining the points

   (i) (5, –3, 8) and (6, –1, 6)

   (ii) (4, 3, –5) and (–2, 1, –8)

4. Find the projection of a line segment joining the points P (4, –2, 5) and Q (2, 1, –3) on the line with direction ratios 6, 2 and 3.

5. Find the projection of a line segment joining the points (2, 1, 3) and (1, 0, –4) on the line joining the points (2, –1, 4) and (0, 1, 5).

33.8 ANGLE BETWEEN TWO LINES WITH GIVEN DIRECTION COSINES

Let OP and OQ be the two lines through the origin O parallel to two lines in space whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively.

Let θ be the angle between OP an OQ and let the co-ordinates of P be (x_1, y_1, z_1).

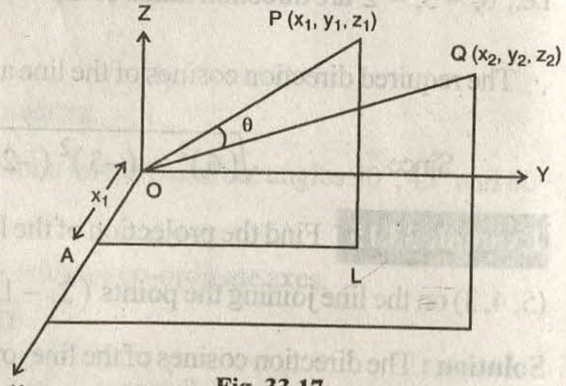

Fig. 33.17

Draw PL perpendicular to XY-plane and LA perpendicular to X-axis. Then the projection of OP on OQ = Sum of projections of OA, AL and LP on OQ.

i.e. $\quad OP \cos\theta = x_1 l_2 + y_1 m_2 + z_1 n_2$ ....(i)

But $\quad x_1 = \text{projection of OP om X-axis} = OP.\, l_1$

Similarly, $\quad y_1 = OP.\, m_1$ and $z_1 = OP.\, n_1$ ....(ii)

Thus, we get, $OP \cos\theta = OP\,(l_1 l_2 + m_1 m_2 + n_1 n_2)$ ....[From (i) and (ii)]

giving $\quad l_1 l_2 + m_1 m_2 + n_1 n_2 = \cos\theta$ ....(iii)

Corollary 1 : If the direction ratio of the lines are a_1, b_1, c_1 and a_2, b_2, c_2 then the angle θ between the two lines is given by

$$\cos\theta = \pm \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Here positive or negative sign is to be taken depending upon θ being acute or obtuse.

Corollary 2 : If OP an OQ are perpendicular to each other, (i.e., if $\theta = 90°$ then

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = \cos 90°$$

$$= 0$$

Corollary 3 : If OP and OQ are parallel, then $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

Since OP || OQ and O is a common point, OP lies on OQ.

Hence $\sin\theta = 0$

Now $\sin^2\theta = 1 - \cos^2\theta = 1 - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$

$$= \left(l_1^2 + m_1^2 + n_1^2\right)\left(l_1^2 + m_1^2 + n_1^2\right) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$$

$$= (l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2$$

and hence $l_1 m_2 - l_2 m_1 = 0$, $m_1 n_2 - m_2 n_1 = 0$ and $n_1 l_2 - n_2 l_1 = 0$

These gives $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Remarks

(a) Two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are

(i) perpendicular if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

(ii) parallel if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

(b) The condition of perpendicularty of two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$\left(\textbf{Hint : } l_1 = \frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}},\ m_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} \text{ and } n_1 = \frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right)$

(c) The condition of parallelism of two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Example 33.16 Find the angle between the two lines whose direction ratios are $-1, 0, 1$ and $0, -1, 1$.

Solution : Let θ be the angle between the given lines.

$$\therefore \qquad \cos\theta = \pm\frac{(-1)\times 0 + 0\times(-1) + 1\times(1)}{\sqrt{(-1)^2+0^2+1^2}\ \sqrt{0^2+(-1)^2+1^2}} = \left(\pm\frac{1}{2}\right)$$

$$\therefore \qquad \theta = 60^\circ \text{ or } 120^\circ$$

Example 33.17 Find the acute angle between the lines whose direction ratios are $5, -12, 13$ and $-3, 4, 5$.

Solution : Let θ be the angle between the two given lines, then

$$\cos\theta = \pm\frac{5(-3)+(-12)4+(13)5}{\sqrt{5^2+(-12)^2+13^2},\ \sqrt{(-3)^2+4^2+5^2}}$$

$$= \pm\frac{-15-48+65}{\sqrt{25+144+169},\ \sqrt{9+16+25}}$$

$$= \pm\frac{2}{\sqrt{169+169},\ \sqrt{50}}$$

$$= \frac{2}{13\sqrt{2}\times 5\sqrt{2}} = \pm\frac{1}{65}$$

Since, θ is acute it is given by $\cos\theta = \frac{1}{65}$

$$\therefore \qquad \theta = \cos^{-1}\left(\frac{1}{65}\right)$$

CHECK YOUR PROGRESS 33.5

1. Find the angle between the lines whose direction ratios are $1, 1, 2$ and $\sqrt{3}-1, -\sqrt{3}-1, 4$.
2. Show that the points A (7, 6, 3), B (4, 10, 1), C (−2, 6, 2) and D (1, 2, 4) are the vertices of a reactangle.
3. By calculating the angle of the triangle with vertices (4, 5, 0), (2, 6, 2) and (2, 3, −1), show that it is an isosceles triangle.
4. Find whether the pair of lines with given direction cosines are parallel or perpendicular.

   (a) $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}; -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}$ (b) $\frac{3}{5}, \frac{4}{5}, \frac{0}{5}\ ;\ -\frac{4}{5}, \frac{3}{5}, \frac{0}{5}$

LET US SUM UP

- For a given point P (x, y, z) in space with reference to reactangular co-ordinate axes, if we draw three planes parallel to the three co-ordinate planes to meet the axes (in A, B and C say), then

  OA = x, OB = y and OC = z where O is the origin.

  Converswly, given any three numbers, x, y and z we can find a unique point in space whose co-ordinates are (x, y, z).

- The distance PQ between the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

  $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

  In particular the distance of P from the origin O is $\sqrt{x_1^2 + y_1^2 + z_1^2}$.

- The co-ordinates of the point which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $l : m$

  (a) internally are $\left(\dfrac{lx_2 + mx_1}{l + m}, \dfrac{ly_2 + my_1}{l + m}, \dfrac{lz_2 + mz_1}{l + m}\right)$

  (b) externally are $\left(\dfrac{lx_2 - mx_1}{l - m}, \dfrac{ly_2 - my_1}{l - m}, \dfrac{lz_2 - mz_1}{l - m}\right)$

  In particular, the co-ordinates of the mid-point of PQ are

  $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

- If l, m and n are the direction cosines of the line, then $l^2 + m^2 + n^2 = 1$.
- The three numbers which are proportional to the direction cosines of a given line are called its direction ratios.
- Direction cosines of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$.
- The projection of the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line with direction cosines l, m and n is $(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$.
- The direction cosines l, m and n of the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by

  $$\frac{x_2 - x_1}{l} = \frac{y_2 - x_1}{m} = \frac{z_2 - x_1}{n} = PQ$$

- The angle θ between two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 is given by $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

  If the lines are

  (a) perpindicular to each other then, $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

  (b) parallel to each other then $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines, then the angle θ between them is given by

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

- The lines will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

SUPPORTIVE WEB SITES

http:// www.wikipedia.org.
http:// mathworld.wolfram.com

TERMINAL EXERCISE

1. Show that the points (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) form an isosceles right-angled triangle.
2. Prove that the points P, Q and R, whose co-ordinates are respectively (3, 2, –4), (5, 4, –6) and (9, 8, –10) are collinear and find the ratio in which Q divides PR.
3. A (3, 2, 0), B (5, 3, 2), C (–9, 6, –3) are three points forming a triangle. AD, the bisector of the angle $\angle BAC$ meets BC at D. Find the co-ordinates of D.

   (**Hint :** D divides BC in the ratio AB : AC)
4. Find the direction cosines of the line joining the point (7, –5, 4) and (5, –3, 8).
5. What are the direction cosines of a line equally inclined to the axes ? How many such lines are there ?
6. Determine whether it is possible for a line to make the angle 45°, 60° and 120° with the co-ordinate axes.

7. Show that the points $(0, 4, 1)$, $(2, 3, -1)$, $(4, 5, 0)$ and $(2, 6, 2)$ are the vertices of a square.

8. Show that the points $(4, 7, 8)$, $(2, 3, 4)$, $(-1, -2, 1)$ and $(1, 2, 5)$ are the vertices of a parallelogram.

9. $A(6, 3, 2)$, $B(5, 1, 4)$, $C(3, -4, 7)$ and $D(0, 2, 5)$ are four points. Find the projections of (i) AB on CD, and (ii) CD on AB.

10. Three vertices of a parallelogram ABCD are $A(3, -4, 7)$, $B(5, 3, -2)$ and $C(1, 2, -3)$. Find the fourth vertex D.

ANSWERS

CHECK YOUR PROGRESS 33.1

1. (a) OX' YZ (b) OXYZ' (c) OX' YZ'
(d) OXY'Z (e) OXYZ'

CHECK YOUR PROGRESS 33.2

1. (a) 7 (b) $2\sqrt{5}$ (c) $\sqrt{3}$

CHECK YOUR PROGRESS 33.3

1. $(1, -3, 2)$ 2. $\left(\frac{13}{5}, \frac{6}{5}, -\frac{13}{5}\right)$; $(5, 18, -17)$

3. $-2:3$ 6. $(2x_2 - x_1, 2y_2 - y_1, 2z_2 - z_1)$

CHECK YOUR PROGRESS 33.4

1. (a) $\pm\frac{3}{\sqrt{14}}, \pm\frac{1}{\sqrt{14}}, \pm\frac{2}{\sqrt{14}}$ (b) $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$

2. (a) $(0, 5, 6)$, $(-3, 0, 6)$ and $(-3, 5, 0)$

(b) $(-3, 0, 0)$, $(0, 5, 0)$ and $(0, 0, 6)$

3. (a) $\pm\frac{1}{3}, \pm\frac{2}{3}, \mp\frac{2}{3}$ (b) $\pm\frac{6}{7}, \pm\frac{2}{7}, \pm\frac{3}{7}$

4. $\frac{30}{7}$. 5. $\frac{7}{3}$.

CHECK YOUR PROGRESS 33.5

1. $\frac{\pi}{3}$ 4. (a) Parallel (b) Perpendicular

TERMINAL EXERCISE

2. 1 : 2 3. $\left(\frac{38}{16}, \frac{57}{16}, \frac{17}{16}\right)$ 4. $\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$ or $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

5. $\pm\frac{1}{\sqrt{3}}, \pm\frac{-1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$; 4 6. Yes.

9. (i) $\frac{13}{7}$ (ii) $\frac{13}{3}$ 10. $(-1, -5, 6)$

34

THE PLANE

Look closely at a room in your house. It has four walls, a roof and a floor. The floor and roof are parts of two parallel planes extending infinitely beyond the boundary. You will also see two pairs of parallel walls which are also parts of parallel planes. Similarly, the tops of tables, doors of rooms etc. are examples of parts of planes.

If we consider any two points in a plane, the line joining these points will lie entirely in the same plane. This is the characteristic of a plane.

Look at Fig.34.1. You know that it is a representation of a rectangular box. This has six faces, eight vertices and twelve edges.

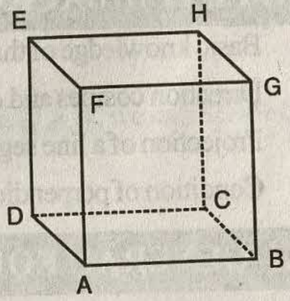

Fig. 34.1

The pairs of opposite and parallel faces are

(i) ABCD and FGHE

(ii) AFED and BGHC

(iii) ABGF and DCHE

and the sets of parallel edges are given below :

(i) AB, DC EH and FG

(ii) AD, BC, GH and FE

(iii) AF, BG, CH and DE

Each of the six faces given above forms a part of the plane, and there are three pairs of parallel planes, denoted by the opposite faces.

In this lesson, we shall establish the general equation of a plane, the equation of a plane passing through three given points, the intercept form of the equation of a plane and the normal form of the equation of a plane. We shall show that a homogeneous equation of second degree in three variables x,y and z represents a pair of planes. We shall also find the equation of a plane bisecting the angle between two planes and area of a triangle in space.

OBJECTIVES

After studying this lesson, you will be able to :

- identify a plane;
- establish the general equation of a plane;
- find the general equation of a plane passing through a given point;
- find the equation of a plane passing through three given points;
- find the equation of a plane in the intercept form and normal form;
- find the angle between two given planes;
- find the equation of a plane bisecting the angle between two given planes; and
- show that the homogeneous equation of second degree in three variables represents a pair of planes.

EXPECTED BACKGROUND KNOWLEDGE

- Basic knowledge of three dimensional geometry.
- Direction cosines and direction ratio of a line.
- Projection of a line segment on another line.
- Condition of perpendicularity and parallelism of two lines in space.

34.1 GENERAL EQUATION OF A PLANE

Recall that a plane is a surface such that if any two points be taken on it, the line joining these two points lies wholly in the plane.

Consider the general equation of first degree in x,y and z

$$ax + by + cz + d = 0 \quad \text{.....(i)}$$

If points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ satisfy the equation (i), then

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{.....(ii)}$$

and $$ax_2 + by_2 + cz_2 + d = 0 \quad \text{.....(iii)}$$

Multiplying (ii) by n and (iii) by m and adding the results,

we get, $a(mx_2 + nx_1) + b(my_2 + ny_1) + c(mz_2 + nz_1) + d(m + n) = 0$

$$\Leftrightarrow \quad a\left(\frac{mx_2 + nx_1}{m+n}\right) + b\left(\frac{my_2 + ny_1}{m+n}\right) + c\left(\frac{mz_2 + nz_1}{m+n}\right) + d = 0 \quad \text{....(iv)}$$

The result (iv) shows that the co-ordinates of the point dividing PQ in the ratio m : n satisfy the equation (i).

$\therefore$ All points lying on the line through P and Q lie on the surface represented by (i)

$\therefore$ The equation (i) represents the general equation of a plane.

34.2 GENERAL EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT

Let $P(x_1, y_1, z_1)$ be a given point and $ax + by + cz + d = 0$ be the given equation of the plane.

Since the plane $ax + by + cz + d = 0$ ...(i)

passes through the point (x_1, y_1, z_1), it implies that

$$ax_1 + by_1 + cz_1 + d = 0 \quad .....(ii)$$

Subtracting (ii) from (i), we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad ....(iii)$$

which is the required equation of a plane passing through the point (x_1, y_1, z_1)

Remark : Can you say, how many planes are represented by (iii) ?

As a,b and c can take any real value, there are infinitely many planes passing through a given point.

Note : In equation (i), there are four constants a,b,c and d. When $d \neq 0$, we can divide (i) by d to get

$$\left(\frac{a}{d}\right)x + \left(\frac{b}{d}\right)y + \left(\frac{c}{d}\right)z + 1 = 0$$

or,

$$Ax + By + Cz + 1 = 0 \quad ....(iv)$$

where $A = \left(\frac{a}{d}\right) B = \left(\frac{b}{d}\right)$ and $C = \left(\frac{c}{d}\right)$

You may note that in (iv), A,B and C are three independent constants.

Example 34.1 Find the equation of a plane passing through the point $(1, -3, 2)$.

Solution : Here $x_1 = 1, y_1 = -3$ and $z_1 = 2$

∴ The required equation of the plane is

$$a(x - 1) + b(y + 3) + c(z - 2) = 0$$

or

$$ax + by + cz + (-a + 3b - 2c) = 0$$

Example 34.2 Find the equation of a plane passing through the point $(3, 0, -2)$

Solution : Here $x_1 = 3, y_1 = 0$ and $z_1 = -2$

∴ The required equation of the plane is

$$a(x - 3) + b(y - 0) + c(z + 2) = 0$$

or

$$ax + by + cz + (-3a + 2c) = 0$$

Example 34.3 Find the equation of a plane passing through a point which divides the line joining the points (2,2,4) and (5,2,1) in the ratio of 1:2 internally.

Solution :

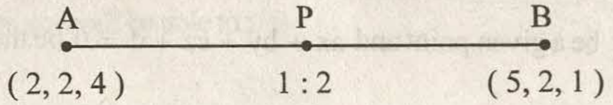

Fig. 34.2

The co-ordinates of P, which divides the join of A and B in the ratio of 1:2 internally are

$$\left(\frac{2\times 2+1\times 5}{2+1},\ \frac{2\times 2+1\times 2}{2+1},\ \frac{2\times 4+1\times 1}{2+1}\right)$$

or $\quad (3, 2, 3)$

$\therefore$ The required equation of the plane is

$$a(x-3)+b(y-2)+c(z-3)=0$$

or

$$ax+by+cz-(3a+2b+3c)=0$$

CHECK YOUR PROGRESS 34.1

1. Find the equation of a plane passing through the origin.
2. Find the equation of a plane passing through the point $(0, 0, -2)$.
3. Find the equation of a plane passing through the point $(5, -7, 3)$.
4. Find the equation of a plane which bisects the line segement joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2).
5. Find the equation of a plane which divides the line joining the points $(1, 2, -3)$ and $(4, 2, -3)$ in the ratio of 1:2 internally.

34.3 EQUATION OF A PLANE PASSING THROUGH THREE GIVEN POINTS

You may recall that the general equation of a plane contains only three independent constants. Hence, a plane can be uniquely determined if it is given to pass through three given non-collinear points.

Let $ax + by + cz + d = 0$ be the equation of the plane and (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be three given points.

The equation of the plane passing through the point (x_1, y_1, z_1) is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \qquad \text{....(i)}$$

If (i) passes through the points (x_2, y_2, z_2) and (x_3, y_3, z_3), then

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \qquad \text{....(ii)}$$

and $$a(x_3 - x_1) + b(y_3 - y_1) + c(z_3 - z_1) = 0 \qquad \text{....(iii)}$$

Eliminating a,b and c from (i), (ii) and (iii), we get the required equation of the plane as

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \qquad \text{....(A)}$$

34.4 EQUATION OF A PLANE IN THE INTERCEPT FORM

Let a, b,c be the lengths of the intercepts made by the plane on the x,y and z axes respectively.

It implies that the plane passes through the points (a,0,0), (0,b,0) and (0,0,c)

Putting $x_1 = a$ $\quad y_1 = 0$ $\quad z_1 = 0$

$x_2 = 0$ $\quad y_2 = b$ $\quad z_2 = 0$

and $x_3 = 0$ $\quad y_3 = 0$ $\quad z_3 = c$ in (A),

we get the required equation of the plane as

$$\begin{vmatrix} x - a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

which on expanding gives $bcx + acy + abz - abc = 0$

or $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{.....(B)}$$

Equation (B) is called the ***Intercept form*** of the equation of the plane.

Example 34.4 Find the equation of the plane passing through the points (0,2,3), (2,0,3) and (2,3,0).

Solution : Using (A), we can write the equation of the plane as

$$\begin{vmatrix} x - 0 & y - 2 & z - 3 \\ 2 - 0 & 0 - 2 & 3 - 3 \\ 2 - 0 & 3 - 2 & 0 - 3 \end{vmatrix} = 0$$

or $$\begin{vmatrix} x & y - 2 & z - 3 \\ 2 & -2 & 0 \\ 2 & 1 & -3 \end{vmatrix} = 0$$

or $\quad x(6 - 0) - (y - 2)(-6) + (z - 3)(2 + 4) = 0$

or $\quad 6x + 6(y - 2) + 6(z - 3) = 0$

or $\quad x+y-2+z-3=0 \quad$ or $\quad x+y+z=5$

Example 34.5 Show that the equation of the plane passing through the points (2,2,0), (2,0,2) and (4,3,1) is $x=y+z$.

Solution : Equation of the plane passing through the point (2,2,0) is

$$a(x-2)+b(y-2)+cz=0 \qquad \text{.....(i)}$$

$\because$ (i) passes through the point (2, 0, 2)

$\therefore \quad a(2-2)+b(0-2)+2c=0$

or $\quad c=b \qquad$ .....(ii)

Again (i) passes through the point (4, 3, 1)

$\therefore \quad a(4-2)+b(3-2)+c=0$

or $\quad 2a+b+c=0 \qquad$ ....(iii)

From (ii) and (iii), we get $2a+2b=0$ or $a=-b$

$\therefore$ (i) becomes

$$-b(x-2)+b(y-2)+bz=0$$

or $\quad -(x-2)+y-2+z=0$

or $\quad y+z-x=0$

or $\quad x=y+z$

Example 34.6 Reduce the equation of the plane $4x-5y+6z-60=0$ to the intercept form. Find its intercepts on the co-ordinate axes.

Solution : The equation of the plane is

$$4x-5y+6z-60=0 \quad \text{or} \quad 4x-5y+6z=60 \qquad \text{....(i)}$$

The equation (i) can be written as $\frac{4x}{60}-\frac{5y}{60}+\frac{6z}{60}=1$ or $\frac{x}{15}+\frac{y}{(-12)}+\frac{z}{10}=1$

which is the interecept form of the equation of the plane and the intercepts on the co-ordinate axes are $15, -12$ and 10 respectively.

CHECK YOUR PROGRESS 34.2

1. Find the equation of the plane passing through the points

   (a) $(2,2,-1),(3,4,2)$ and $(7,0,6)$

   (b) $(2,3,-3),(1,1,-2)$ and $(-1,1,4)$

   (c) $(2,2,2),(3,1,1)$ and $(6,-4,-6)$

2. Show that the equation of the plane passing through the points $(3,3,1)$, $(-3,2-1)$ and $(8,6,3)$ is $4x+2y-13z=5$

3. Find the equation of a plane whose intercepts on the coordinate axes are 2,3 and 4 respectively.

4. Find the intercepts made by the plane $2x+3y+4z=24$ on the co-ordinate axes.

5. Show that the points $(-1,4,-3)$, $(3,2,-5)$, $(-3,8,-5)$ and $(-3,2,1)$ are coplanar.

34.5 EQUATION OF A PLANE IN THE NORMAL FORM

Let ON be perpendicular from the origin O to the plane ABC and let $P(x,y,z)$ be any point on the plane. Let PL be drawn perpendicular to XY plane and LM be perpendicular to OX so that $OM=x$, $ML=y$ and $PL=z$. The sum of the projections of OM, ML and LP on ON is equal to the projection of OP on ON.

If l, m and n are the direction cosines of ON and p is the length of the perpendicular ON, then $lx+my+nz=p$ ....(i)

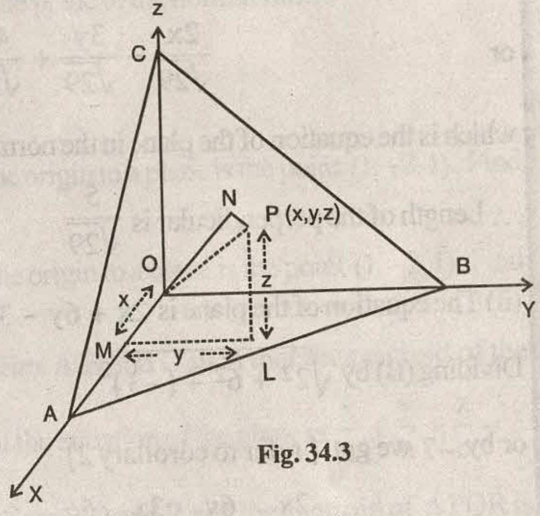

Fig. 34.3

(i) is called the normal form of the equation of the plane.

We know that $l=\cos\alpha$, $m=\cos\beta$ and $n=\cos\gamma$, where α, β and γ are angles made by ON with the positive directions of x, y and z axes respectively.

∴ (i) can alternatively be written as

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p$$

Corollary 1 : By comparing the general equation of first degree in x, y, z, i.e., $ax+by+cz+d=0$ with $lx+my+nz=p$, we see the direction cosines of normal to the plane $ax+by+cz+d=0$ are proportional to a, b and c and are equal to

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{b}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{c}{\sqrt{a^2+b^2+c^2}},$$

the positive or negative sign to be taken according as d is positive or negative as p, by convention, is taken to be positive.

Also, the length of the perpendicular from the origin to the plane is $\frac{|-d|}{\sqrt{a^2+b^2+c^2}}$

Corollary 2 : The general equation of a plane $ax+by+cz+d=0$ is reduced to normal form by dividing each term by $-\sqrt{a^2+b^2+c^2}$ or, $\sqrt{a^2+b^2+c^2}$ according as d is positive or negative.

Example 34.7 Reduce each of the following equations of the plane to the normal form :

(i) $2x - 3y + 4z - 5 = 0$ (ii) $2x + 6y - 3z + 5 = 0$

Find the length of perpendicular from origin upon the plane in both the cases.

Solution : (i) The equation of the plane is $2x - 3y + 4z - 5 = 0$ ....(A)

Dividing (A) by $\sqrt{2^2 + (-3^2) + 4^2}$ or, by $\sqrt{29}$

we get, $$\frac{2x}{\sqrt{29}} - \frac{3y}{\sqrt{29}} + \frac{4z}{\sqrt{29}} - \frac{5}{\sqrt{29}} = 0$$

or $$\frac{2x}{\sqrt{29}} - \frac{3y}{\sqrt{29}} + \frac{4z}{\sqrt{29}} = \frac{5}{\sqrt{29}}$$

which is the equation of the plane in the normal form.

$\therefore$ Length of the perpendicular is $\frac{5}{\sqrt{29}}$

(ii) The equation of the plane is $2x + 6y - 3z + 5 = 0$ .....(B)

Dividing (B) by $\sqrt{2^2 + 6^2 + (-3)^2}$

or by -7 we get, [refer to corollary 2]

$$-\frac{2x}{7} - \frac{6y}{7} + \frac{3z}{7} - \frac{5}{7} = 0 \text{ or } -\frac{2x}{7} - \frac{6y}{7} + \frac{3z}{7} = \frac{5}{7}$$

which is the required equation of the plane in the normal form.

$\therefore$ Length of the perpendicular from the origin upon the plane is $\frac{5}{7}$

Example 34.8 The foot of the perpendicular drawn from the origin to the plane is $(4, -2, -5)$. Find the equation of the plane.

Solution : Let P be the foot of perpendicular drawn from origin O to the plane.

Then P is the point $(4, -2, -5)$.

The equation of a plane through the point $P(4, -2, -5)$ is

$$a(x-4) + b(y+2) + c(z+5) = 0 \qquad .....(i)$$

Now OP $\perp$ plane and direction cosines of OP are proportional to

$$4-0, -2-0, -5-0$$

i.e. $$4, -2, -5.$$

Substituting $4, -2$ and -5 for a, b and c in (i), we get

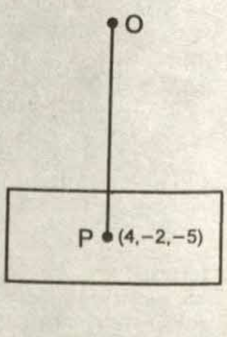

Fig. 34.4

$$4(x-4)-2(y+2)-5(z+5)=0$$

or
$$4x-16-2y-4-5z-25=0$$

or
$$4x-2y-5z=45$$

which is the required equation of the plane.

CHECK YOUR PROGRESS 34.3

1. Reduce each of the following equations of the plane to the normal form :

   (i) $4x+12y-6z-28=0$

   (ii) $3y+4z+3=0$

2. The foot of the perpendicular drawn from the origin to a plane is the point $(1,-3,1)$. Find the equation of the plane.

3. The foot of the perpendicular drawn from the origin to a plane is the point $(1,-2,1)$. Find the equation of the plane.

4. A plane meets the co-ordinate axes in points A,B and C such that the centroid of the triangle ABC is the point (a,b,c). Prove that the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$

5. A plane meets the coordinate axes at the points P, Q and R and the centroid of ΔPQR is the point $(3,4,-6)$. Find the equation of the plane.

34.6 ANGLE BETWEEN TWO PLANES

Let the two planes p_1 and p_2 be given by

$$a_1x+b_1y+c_1z+d_1=0 \quad \text{....(i)}$$

and $a_2x+b_2y+c_2z+d_2=0$ ....(ii)

Let the two planes intersect in the line l and let n_1 and n_2 be normals to the two planes. Let θ be the angle between two planes.

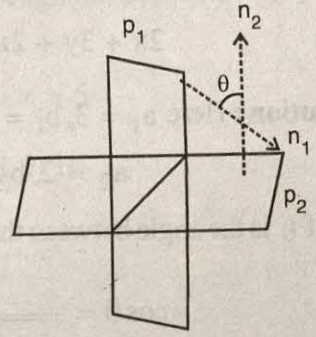

Fig. 34.5

$\therefore$ The direction cosines of normals to the two planes are

$$\pm\frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}},\ \pm\frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}},\ \pm\frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$$

and
$$\pm\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}},\ \pm\frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}},\ \pm\frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}},$$

$\therefore$ $\cos\theta$ is given by $\cos\theta = \pm \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$

where the sign is so chosen that $\cos\ \theta$ is positive

Corollary 1 :

When the two planes are perpendicular to each other then $\theta = 90°$ i.e., $\cos\theta = 0$

$\therefore$ The condition for two planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$ to be perpendicular to each other is

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Corollary 2 :

If the two planes are parallel, then the normals to the two planes are also parallel

$$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$\therefore$ The condition of parallelism of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and

$$a_2x + b_2y + c_2z + d_2 = 0 \text{ is } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

This implies that the equations of two parallel planes differ only by a constant. Therefore, any plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is a constant.

Example 34.9 Find the angle between the planes

$$3x + 2y - 6z + 7 = 0 \qquad \text{......(i)}$$

and

$$2x + 3y + 2z - 5 = 0 \qquad \text{......(ii)}$$

Solution : Here $a_1 = 3, b_1 = 2, c_1 = -6$

and $a_2 = 2, b_2 = 3, c_2 = 2$

$\therefore$ If θ is the angle between the planes (i) and (ii), then

$$\cos\theta = \frac{3.2 + 2.3 + (-6).2}{\sqrt{3^2 + 2^2 + (-6)^2}\ \sqrt{2^2 + 3^2 + 2^2}} = 0$$

$\therefore \theta = 90°$

Thus the two planes given by (i) and (ii) are perpendicular to each other.

Example 34.10 Find the equation of the plane parallel to the plane $x - 3y + 4z - 1 = 0$ and passing through the point $(3, 1, -2)$.

Solution : Let the equation of the plane parallel to the plane

$$x - 3y + 4z - 1 = 0 \text{ be } x - 3y + 4z + k = 0 \quad \text{..... (i)}$$

Since (i) passes through the point $(3, 1, -2)$, it should satisfy it

$\therefore \quad 3 - 3 - 8 + k = 0 \quad \text{or} \quad k = 8$

$\therefore$ The required equation of the plane is $x - 3y + 4z + 8 = 0$

Example 34.11 Find the equation of the plane passing through the points $(-1, 2, 3)$ and $(2, -3, 4)$ and which is perpendicular to the plane $3x + y - z + 5 = 0$

Solution : The equation of any plane passing through the point $(-1, 2, 3)$ is

$$a(x + 1) + b(y - 2) + c(z - 3) = 0 \quad \text{.....(i)}$$

Since the point $(2, -3, 4)$ lies on the plane (i)

$$\therefore \quad 3a - 5b + c = 0 \quad \text{....(ii)}$$

Again the plane (i) is perpendicular to the plane $3x + y - z + 5 = 0$

$$\therefore \quad 3a + b - c = 0 \quad \text{....(iii)}$$

From (ii) and (iii), by cross multiplication method, we get,

$$\frac{a}{4} = \frac{b}{6} = \frac{c}{18} \quad \text{or} \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{9}$$

Hence the required equation of the plane is

$$2(x + 1) + 3(y - 2) + 9(z - 3) = 0 \quad \text{....[From (i)]}$$

or

$$2x + 3y + 9z = 31$$

Example 34.12 Find the equation of the plane passing through the point $(2, -1, 5)$ and perpendicular to each of the planes

$$x + 2y - z = 1 \qquad \text{and } 3x - 4y + z = 5$$

Solution : Equation of a plane passing through the point $(2, -1, 5)$ is

$$a(x - 2) + b(y + 1) + c(z - 5) = 0 \quad \text{.....(i)}$$

As this plane is perpendicular to each of the planes

$$x + 2y - z = 1 \qquad \text{and} \qquad 3x - 4y + z = 5$$

We have $\quad a.1 + b.2 + c.(-1) = 0$

and $\quad a.3 + b.(-4) + c.(1) = 0$

or $\quad a + 2b - c = 0 \quad \text{.........(ii)}$

$\quad 3a - 4b + c = 0 \quad \text{........(iii)}$

From (ii) and (iii), we get

$$\frac{a}{2-4}=\frac{b}{-3-1}=\frac{c}{-4-6}$$

or $$\frac{a}{-2}=\frac{b}{-4}=\frac{c}{-10} \quad \text{or} \quad \frac{a}{1}=\frac{b}{2}=\frac{c}{5}=\lambda \text{ (say)}$$

$\therefore$ $$a=\lambda,\ b=2\lambda \text{ and } c=5\lambda$$

Substituting for a, b and c in (i), we get

$$\lambda(x-2)+2\lambda(y+1)+5\lambda(z-5)=0$$

or $$x-2+2y+2+5z-25=0$$

or $$x+2y+5z-25=0$$

which is the required equation of the plane.

CHECK YOUR PROGRESS 34.4

1. Find the angle between the planes

   (i) $2x-y+z=6$ and $x+y+2z=3$

   (ii) $3x-2y+z+17=0$ and $4x+3y-6z+25=0$

2. Prove that the following planes are perpendicular to each other.

   (i) $x+2y+2z=0$ and $2x+y-2z=0$

   (ii) $3x+4y-5z=9$ and $2x+6y+6z=7$

3. Find the equation of the plane passing through the point $(2,3,-1)$ and parallel to the plane $2x+3y+6z+7=0$

4. Find the equation of the plane through the points $(-1,1,1)$ and $(1,-1,1)$ and perpendicular to the plane $x+2y+2z=5$

5. Find the equation of the plane which passes through the origin and is perpendicular to each of the planes $x+2y+2z=0$ and $2x+y-2z=0$

34.7 DISTANCE OF A POINT FROM A PLANE

Let the equation of the plane in normal form be

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p \text{ where } p>0 \qquad \text{.......(i)}$$

Case I : Let the point $P(x',y',z')$ lie on the same side of the plane in which the origin lies.

Let us draw a plane through point P parallel to plane (i).Its equation is

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p' \qquad \text{.....(ii)}$$

where p' is the length of the perpendicular drawn from origin upon the plane given by (ii). Hence the perpendicular distance of P from plane (i) is $p-p'$

As the plane (ii) passes through the point (x', y', z'),

$$x'\cos\alpha + y'\cos\beta + z'\cos\gamma = p'$$

$\therefore$ The distance of P from the given plane is

$$p - p' = p - (x'\cos\alpha + y'\cos\beta + z'\cos\gamma)$$

Case II : If the point P lies on the other side of the plane in which the origin lies, then the distance of P from the plane (i) is,

$$p' - p = x'\cos\alpha + y'\cos\beta + z'\cos\gamma - p$$

Note : If the equation of the plane be given as $ax + by + cz + d = 0$, we have to first convert it into the normal form, as discussed before, and then use the above formula.

34.8 EQUATION OF A PLANE BISECTING THE ANGLE BETWEEN TWO PLANES

Let the equations of the planes be $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

If $P(x_1, y_1, z_1)$ be any point on the plane bisecting the angle between two given planes, then the lengths of the perpendiculars drawn from P to them should be equal.

$$\therefore \quad \frac{a_1x_1 + b_1y_1 + c_1z_1 + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x_1 + b_2y_1 + c_2z_1 + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

according as the two perpendiculars are of the same sign or of different sign.

$\therefore$ The equation

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \qquad \text{......(i)}$$

is satisfied by co-ordinates of any point on the planes bisecting the angles between the two given planes.

$\therefore$ (i) represents the equations of the planes which bisect the angles between the two given planes.

Note : You may note that we get two planes bisecting the angles between the given planes.

Example 34.13 Find the distance of the point (1,2,3) from the plane $3x - 2y + 5z + 17 = 0$

Solution : Required distance $= \dfrac{3.1 - 2.2 + 5.3 + 17}{\sqrt{3^2 + (-2)^2 + 5^2}} = \dfrac{31}{\sqrt{38}}$ units.

Example 34.14 Find the distance between the planes

$$x - 2y + 3z - 6 = 0$$

and

$$2x - 4y + 6z + 17 = 0$$

Solution : The equations of the planes are

$$x - 2y + 3z - 6 = 0 \quad \text{....(i)}$$

$$2x - 4y + 6z + 17 = 0 \quad \text{....(ii)}$$

Here $$\frac{1}{2} = \frac{(-2)}{(-4)} = \frac{3}{6}$$

$\therefore$ Planes (i) and (ii) are parallel

Any point on plane (i) is (6, 0, 0)

$\therefore$ Distance between planes (i) and (ii) = Distance of point (6,0,0) from (ii)

$$= \frac{2 \times 6 - 4.0 + 6.0 + 17}{\sqrt{(2)^2 + (-4)^2 + 6^2}}$$

$$= \frac{29}{\sqrt{56}} \text{ units} = \frac{29}{2\sqrt{14}} \text{ units}$$

Example 34.15 Find the equation of the planes which bisect the angles between the planes $3x - 2y + 6z + 8 = 0$ and $2x - y + 2z + 3 = 0$

Solution : The required equations of the bisector planes are

$$\frac{3x - 2y + 6z + 8}{\sqrt{3^2 + (-2)^2 + (6)^2}} = \pm \frac{2x - y + 2z + 3}{\sqrt{2^2 + (-1)^2 + 2^2}}$$

or $$\frac{3x - 2y + 6z + 8}{7} = \pm \frac{2x - y + 2z + 3}{3}$$

or $$9x - 6y + 18z + 24 = +(14x - 7y + 14z + 21)$$

or $$9x - 6y + 18z + 24 = -(14x - 7y + 14z + 21)$$

$$\Rightarrow \quad 5x - y - 4z = 3 \quad \text{or} \quad 23x - 13y + 32z + 45 = 0$$

CHECK YOUR PROGRESS 34.5

1. Find the distance of the point

   (i) $(2, -3, 1)$ from the plane $5x - 2y + 3z + 11 = 0$

   (ii) $(3, 4, -5)$ from the plane $2x - 3y + 3z + 27 = 0$

2. Find the distance between planes

   $3x + y - z - 7 = 0$ and $6x + 2y - 2z + 11 = 0$

3. Find the equations of the planes bisecting the angles between the planes

   (i) $2x + y - 2z = 4$ and $2x - 3y + 6z + 2 = 0$

(ii) $3x - 4y + 12z = 26$ and $x + 2y - 2z = 9$

(iii) $x + 2y + 2z - 9 = 0$ and $4x - 3y + 12z + 13 = 0$

34.9 HOMOGENEOUS EQUATION OF SECOND DEGREE REPRESENTING TWO PLANES

Let the equations of two planes be

$$A \equiv a_1x + b_1y + c_1z + d_1 = 0 \qquad \text{....(i)}$$

and

$$B \equiv a_2x + b_2y + c_2z + d_2 = 0 \qquad \text{....(ii)}$$

Consider the equation $AB = 0$

i.e, $(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0$ ....(iii)

The co-ordinates of all points satisfying (i) or (ii) satisfy the equation (iii)

i.e., all points of $A = 0$ or, $B = 0$ lie on the surface represented by the equation $AB = 0$

Again $AB = 0$ is a second degree equation in x, y and z.

$\therefore$ A second degree equation in x, y and z represents a pair of planes only if it can be resolved into two linear factors (each factor equated to zero representing a plane)

Note : We give below an important result regarding the factorisation of a general homogeneous equation of second degree in x, y and z.

viz. $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gxz = 0$ ....(A)

(A) is factorisable into two linear factors if and only if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Also, the angle θ between the two planes is given by

$$\tan\theta = \frac{2\sqrt{f^2 + g^2 + h^2 - ab - bc - ca}}{a + b + c}$$

The proofs of these results are beyond the scope of this lesson.

Example 34.16 Prove that the equation

$$8x^2 - 3y^2 - 10z^2 + 10xy + 17yz + 2xz = 0$$

represents a pair of planes through the origin. Also find the angle between the two planes.

Solution : Here $a = 8$, $b = -3$, $c = -10$, $h = 5$, $f = \frac{17}{2}$ and $g = 1$

$\therefore \quad abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 8(-3)(-10) + 2\left(\frac{17}{2}\right)(1)(5) - 8\left(\frac{17}{2}\right)^2 + 3(1)^2 + 10(5)^2$$

$$= 240 + 85 - 578 + 3 + 250$$

$$= 578 - 578 = 0$$

∴ The given equation

$8x^2 - 3y^2 - 10z^2 + 10xy + 17yz + 2xz = 0$ represent a pair of planes through the origin.

Also, angle θ between the planes is given by

$$\tan\theta = \frac{2\sqrt{f^2 + g^2 + h^2 - ab - bc - ca}}{a + b + c}$$

$$\tan\theta = \frac{2\sqrt{\frac{289}{4} + 1 + 25 + 24 - 30 + 80}}{-5} = -\frac{2}{5}\sqrt{\frac{689}{4}}$$

$$= -\frac{2}{5}\sqrt{\frac{689}{4}}$$

$$\therefore \qquad \theta = \tan^{-1}\left[-\frac{2}{5}\sqrt{\frac{689}{4}}\right]$$

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CHECK YOUR PROGRESS 34.6

1. Prove that the equation $2x^2 - 6y^2 - 12z^2 + 18yz + 2xz + xy = 0$ represents a pair of planes. Find also the angle between the planes.

2. Show that the equation $6x^2 + 4y^2 - 10z^2 + 3yz + 4xz - 11xy = 0$ represents a pair of planes and find the angle between the planes.

APPENDIX

A special method for factorising a general homogeneous polynomial of second degree in x,y and z.

For example, let us consider the polynomial

$$8x^2 - 3y^2 - 10z^2 + 10xy + 17yz + 2xz \qquad \text{...(i)}$$

Putting z = 0 in (i), we have $8x^2 - 3y^2 + 10xy$

or $8x^2 + 12xy - 2xy - 3y^2$

or $4x(2x + 3y) - y(2x + 3y)$

or $(4x - y)(2x + 3y)$ ....(A)

Putting y = 0 in (i)

we have $8x^2 + 2xz - 10z^2$

or $8x^2 + 10xz - 8xz - 10z^2$

or $2x(4x + 5z) - 2z(4x + 5z)$

or $(2x - 2z)(4x + 5z)$ ....(B)

Let x = 0 in (i), we get

$-3y^2 - 10z^2 + 17yz$

or $3y^2 - 17yz + 10z^2$

or $3y^2 - 2yz - 15yz + 10z^2$

or $y(3y - 2z) - 5z(3y - 2z)$

or $(3y - 2z)(y - 5z)$ ....(C)

Combining (A), (B), (C), we get the factors of (i) as

$(4x - y + 5z)$ and $(2x + 3y - 2z)$

$$\therefore \quad 8x^2 - 3y^2 - 10z^2 + 10xy + 17yz + 2xz \equiv (4x - y + 5z)(2x + 3y - 2z)$$

This method is applicable for all such homogeneous polynomials of second degree in x,y and z.

Note :

(i) Look at the factor (2x+3y) in (A) and the factor $(2x - 2z)$ in (B). As 2x is common between the two, the complete factor is $(2x + 3y - 2z)$. Similarly, looking at $(4x - y)$ in (A) and $(4x + 5z)$ in (B), we get the complete factor as $(4x - y + 5z)$.

(ii) The equations of the two planes represented by the equation

$8x^2 - 3y^2 - 10z^2 + 10xy + 17yz + 2xz = 0$ are

$$4x - y + 5z = 0 \qquad \text{and} \qquad 2x + 3y - 2z = 0.$$

(iii) You are advised to find the equations of the pairs of planes in **Check Your Progress 34.6** using the method as shown in the appendix.

34.10 AREA OF A TRIANGLE

Let $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ be the direction cosines of the normal to the plane whose area is A. Then the

(i) projection of the area A, say A_1, on yz-plane is $A\cos\alpha$

i.e., $$A_1 = A\cos\alpha$$

(ii) Similarly A_2, the projection of the area A on xz-plane and A_3, the projection of A on xy-plane are given by

$$A_2 = A\cos\beta \text{ and } A_3 = A\cos\gamma$$

$$\therefore \quad A_1^2 + A_2^2 + A_3^3 = A^2\cos^2\alpha + A^2\cos^2\beta + A^2\cos^2\gamma$$

$$= A^2\left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right)$$

$$= A^2 \quad \left(\text{since } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1\right)$$

We shall use this result in finding the area of a triangle.

Let the vertices of the triangle ABC be $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ respectively and let the area of the triangle be Δ. Let the projections of Δ on the co-ordinates planes be Δ_{xy}, Δ_{yz} and Δ_{zx} respectively.

The projection of A, B and C on xy-plane are

$$(x_1, y_1, 0), (x_2, y_2, 0) \text{ and } (x_3, y_3, 0)$$

$$\therefore \quad \Delta_{xy} = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad \text{...........(A)}$$

[Recall that you have found the area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) as given in (A) in two dimensional co-ordinate geometry]

Similarly, $$\Delta_{yz} = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \text{ and } \Delta_{zx} = \frac{1}{2}\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}$$

$$\therefore \quad \Delta^2 = \left(\Delta_{xy}\right)^2 + \left(\Delta_{yz}\right)^2 + \left(\Delta_{xz}\right)^2$$

$$= \frac{1}{4}\left[\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2\right]$$

which gives required area of the triangle in space.

Example 34.17 Find the area of the triangle with vertices (1,5,2), (1, 2, –3) and (3, 2, –1)

Solution : Here

$$\Delta^2{}_{xy} = \frac{1}{4}\begin{vmatrix} 1 & 5 & 1 \\ 1 & 2 & 1 \\ 3 & 2 & 1 \end{vmatrix}^2 = \frac{1}{4}[1(2-2)-5(1-3)+1(2-6)]^2$$

$$= \frac{1}{4}(10-4)^2 = 9$$

$$\Delta^2{}_{yz} = \frac{1}{4}\begin{vmatrix} 5 & 2 & 1 \\ 2 & -3 & 1 \\ 2 & -1 & 1 \end{vmatrix}^2 = \frac{1}{4}[5(-3+1)-2(2-2)+1(-2+6)]^2$$

$$= \frac{1}{4}(-10+4)^2 = 9$$

$$\Delta^2{}_{zx} = \frac{1}{4}\begin{vmatrix} 2 & 1 & 1 \\ -3 & 1 & 1 \\ -1 & 3 & 1 \end{vmatrix}^2 = \frac{1}{4}[2(1-3)-1(-3+1)+1(-9+1)]^2$$

$$= \frac{1}{4}(-4+2-8)^2 = 25$$

$\therefore$ $\Delta^2 = 9 + 9 + 25 = 43$

$\therefore$ $\Delta = \sqrt{43}$ sq. units.

CHECK YOUR PROGRESS 34.7

1. Find the area of the triangles with vertices

   (i) $(2,-5,3),(1,-7,4),(2,-3,5)$

   (ii) $(3,0,1),(4,-1,-1),(3,-2,-2)$

   (iii) $(a,0,0),(0,b,0),(0,0,c)$

LET US SUM UP

- A plane is a surface such that if any two points are taken on it, the line joining these two points lies wholly in the plane.

- The general equation of a plane is $ax + by + cz + d = 0$
- The equation of a plane passing through a given point (x_1, y_1, z_1) is
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$
- There are infinite number of planes passing through a given point.
- The equation $Ax + By + Cz + 1 = 0$ of the plane, contains three independent constants.
- The equation of a plane passing through three points
$(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3)
$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$
- Equation of a plane in the intercept from is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
where a,b and c are intercepts made by the plane on x,y and z axes respectively.
- Equation of a plane in the normal form is $lx + my + nz = p$
where l, m, n are the direction cosines of normal to the plane and p is the length of the perpendicular from the origin to the plane.
- Angle θ between two planes $a_1x + b_1y + c_1z + d_1 = 0$
and $a_2x + b_2y + c_2z + d_2 = 0$ is given by
$$\cos\theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$
- Two planes are perpendicular to each other if and only if
$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$
- Two planes are parallel if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- Distance of a point (x', y', z') from a plane
$$x\cos\alpha + y\cos\beta + z\cos\gamma = p \text{ is}$$
$|p - (x'\cos\alpha + y'\cos\beta + z'\cos\gamma)|$, where the point (x', y', z') lies on the same side of the plane in which the origin lies.
- Equations of the planes bisecting the angles between two planes
$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are given by
$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

- A homogeneous equation of second degree

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gxz = 0$$

represents a pair of planes if and only if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

- The square of the area of a triangle with vertices $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) is

$$= \frac{1}{4}\left[\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2\right]$$

SUPPORTIVE WEB SITES

http:// www.wikipedia.org.
http:// mathworld.wolfram.com

TERMINAL EXERCISE

1. Find the equation of a plane passing through the point $(-2, 5, 4)$
2. Find the equation of a plane which divides the line segment joining the points $(2, 1, 4)$ and $(2, 6, 4)$ internally in the ratio of 2 : 3.
3. Find the equation of the plane through the points $(1, 1, 0)$, $(1, 2, 1)$ and $(-2, 2, -1)$.
4. Show that the four points $(0, -1, -1), (4, 5, 1), (3, 9, 4)$ and $(-4, 4, 4)$ are coplanar. Also find the equation of the palne in which they lie.
5. The foot of the perpendicular drawn from $(1, -2, -3)$ to a plane is $(3, 2, -1)$. Find the equation of the plane.
6. Find the angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$
7. Prove that the planes $3x - 5y + 8z - 2 = 0$ and $12x - 20y + 32z + 9 = 0$ are parallel.
8. Determine the value of k for which the planes $3x - 2y + kz - 1 = 0$ and $x + ky + 5z + 2 = 0$ may be perpendicular to each other.
9. Find the distance of the point $(3, 2, -5)$ from the plane $2x - 3y - 5z = 7$

10. Find the equation of the planes that bisect angles between the planes

$$2x - y + 2z + 3 = 0 \text{ and } 3x - 2y + 6z + 8 = 0$$

11. Prove that the equation $6x^2 - 12y^2 + 4z^2 + xy + 13yz - 14xz = 0$ represents a pair of planes through the origin. Find also the angle between the two planes and the equations of the two planes.

12. Find the area of the triangle with vertices $(1,2,1),(3,4,-2)$ and $(4,2,-1)$

ANSWERS

CHECK YOUR PROGRESS 34.1

1. $ax + by + cz = 0$ 2. $ax + by + c(z+2) = 0$

3. $a(x-5) + b(y+7) + c(z-3) = 0$

4. $a\left(x - \frac{x_1 + x_2}{2}\right) + b\left(y - \frac{y_1 + y_2}{2}\right) + c\left(z - \frac{z_1 + z_2}{2}\right) = 0$

5. $a(x-2) + b(y-2) + c(z+3) = 0$

CHECK YOUR PROGRESS 34.2

1. (a) $5x + 2y - 3z - 17 = 0$ (b) $3x - y + z = 0$

(c) $x + 2y - z = 4$

3. $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$

4. Intercepts are 12, 8 and 6 on the co-ordinate axes x,y and z respectively.

CHECK YOUR PROGRESS 34.3

1. (i) $\frac{4x}{14} + \frac{12y}{14} - \frac{6z}{14} = 2$ (ii) $-\frac{3}{5}y - \frac{4}{5}z = \frac{3}{5}$

2. $x - 3y + z - 11 = 0$ 3. $x - 2y + z - 6 = 0$

5. $4x + 3y - 2z = 36$

CHECK YOUR PROGRESS 34.4

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{2}$

3. $2x + 3y + 6z = 7$

4. $2x + 2y - 3z + 3 = 0$

5. $2x - 2y + z = 0$

CHECK YOUR PROGRESS 34.5

1. (i) $\frac{30}{\sqrt{38}}$ units (ii) $\frac{6}{\sqrt{22}}$ units.

2. $\frac{25}{2\sqrt{11}}$ units.

3. (i) $10x - y + 2z - 11 = 0$; $4x + 8y - 16z - 17 = 0$

(ii) $4x + 38y - 62z - 39 = 0$; $22x + 14y + 10z - 195 = 0$

(iii) $25x + 17y + 62z - 78 = 0$; $x + 35y - 10z - 156 = 0$

CHECK YOUR PROGRESS 34.6

1. $\cos^{-1}\left(\frac{16}{21}\right)$ 3. $\frac{\pi}{2}$

CHECK YOUR PROGRESS 34.7

1. (i) $\sqrt{11}$ sq. units (ii) $\frac{\sqrt{14}}{2}$ sq. units

(iii) $\frac{1}{2}\sqrt{(a^2b^2 + b^2c^2 + c^2a^2)}$ sq. units.

TERMINAL EXERCISE

1. $a(x+2) + b(y-5) + c(z-4) = 0$

2. $a(x-2) + b(y-3) + c(z-4) = 0$

3. $2x + 3y - 3z - 5 = 0$

4. $5x - 7y + 11z + 4 = 0$ 5. $x + 2y + z = 6$

6. $\frac{\pi}{3}$ 8. $k = -1$ 9. $\frac{18}{\sqrt{38}}$

10. $5x - y - 4z - 3 = 0$, $23x - 13y + 32z + 45 = 0$

11. $3x - 4y - z = 0$; $2x + 3y - 4z = 0$; $\cos^{-1}\left[\frac{-2}{\sqrt{26}\sqrt{29}}\right]$

12. $\frac{1}{2}\sqrt{77}$ sq units.

35

THE STRAIGHT LINE

In Fig. 35.1, we see a rectangular box having six faces, which are parts of six planes. In the figure, ABCD and EFGH are parallel planes. Similarly, ADGH and BCFE are parallel planes and so are ABEH and CFGD. Two planes ABCD and CFGD intersect in the line CD. Similarly, it happens with any two adjacent planes. Also two edges, say AB and AH meet in the vertex A. It also happens with any two adjacent edges. We can see that the planes meet in lines and the edges meet in vertices.

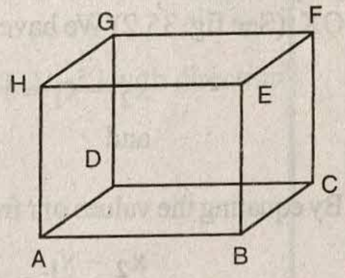

Fig. 35.1

In this lesson, we will study the equations of a line in space in symmetric form, reducing the general equation of a line into symmetric form, finding the perpendicular distance of a point from a line and finding the angle between a line and a plane. We will also establish the condition of coplanarity of two lines.

OBJECTIVES

After studying this lesson, you will be able to :

- find the equations of a line in space in symmetric form;
- convert the general equations of a line into symmetric form;
- find the perpendicular distance of a point from a line;
- find the angle between a line and a plane; and
- find the condition of coplanarity of two lines.

EXPECTED BACKGROUND KNOWLEDGE

- Basic knowledge of three dimensional geometry.
- Direction cosines/ratios of a line and projection of a line segment on another line.
- Condition of parallelism and perpendicularity of two lines.

- General equation of a plane.
- Equations of a plane in different forms.
- Angle between two planes.

35.1 EQUATIONS OF A LINE IN SYMMETRIC FORM

Notes

Let us consider a line segment whose end points are $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Let the distance between points P_1 and P_2 be r and direction cosines of P_1P_2 be $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. Draw perpendiculars P_1L and P_2M from the points P_1 and P_2 on the XY-plane and draw perpendiculars from L and M on OY and OX. (See fig. 35.2) We have

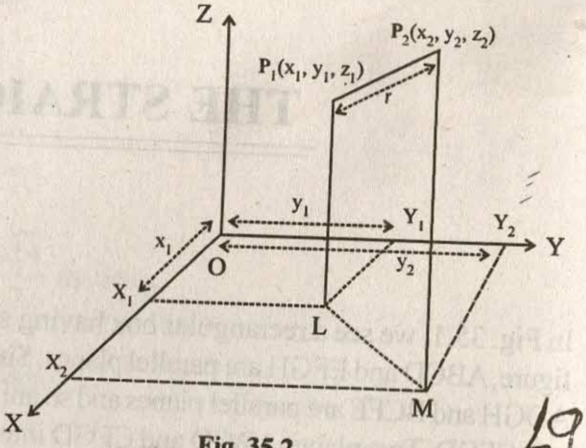

Fig. 35.2

L9

$$\left.\begin{array}{l} x_2 - x_1 = r\cos\alpha,\ y_2 - y_1 = r\cos\beta \\ \text{and} \qquad\qquad z_2 - z_1 = r\cos\gamma \end{array}\right\} \qquad \text{.....(i)}$$

By equating the values of r from (i), we have

$$\frac{x_2 - x_1}{\cos\alpha} = \frac{y_2 - y_1}{\cos\beta} = \frac{z_2 - z_1}{\cos\gamma}$$

That is the projections of a line-segment on the coordinate axes are proportional to the direction cosines of the line-segment.

If the coordinates of P_1 and P_2 are given, the direction cosines of P_1P_2 can be obtained. Even if only one-point $P_1(x_1, y_1, z_1)$ and the direction cosines are given, the line can be determined. So if $P_2(x_2, y_2, z_2)$ is taken as an arbitrary point $P(x, y, z)$, we can write the equations of the line through $P_1(x_1, y_1, z_1)$ with direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ as

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma} = r \text{ (say)} \qquad \text{.....(ii)}$$

If the direction cosines of the line are l, m and n, then equations (ii) can be written as

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \qquad \text{.....(A)}$$

Equations (A) are called the equations of the line in symmetric form.

You may note that when the equations of the line are in symmetric form, they are two linearly independent equations of planes, which indicates that a line is the intersection of two planes.

Corollaries :(i) The co-ordinates of any point P on the line whose direction cosines are l, m and n and which passes through the point (x_1, y_1, z_1) are $(x_1 + lr, y_1 + mr, z_1 + nr)$ where r is the distance of the point P from the fixed point (x_1, y_1, z_1).

(ii) The equations of a line through (x_1, y_1, z_1) with direction ratios a,b,c will be

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r \qquad .....(B)$$

In this case, r does not represent the actual distance of the point P from the point (x_1, y_1, z_1).

(iii) Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, be two points in space. The direction cosines of P_1P_2 are proportional to $(x_2 - x_1), (y_2 - y_1)$ and $(z_2 - z_1)$. If $P(x, y, z)$ be any point on the line P_1P_2, then the equations of the line passing through the points P_1 and P_2 are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad ....(C)$$

(C) are called the ***two-point form*** of the equations of a line.

Example 35.1 Find the equations of the line through the point $(1, 2, -3)$ with direction cosines

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

Solution : The equations of the line are (Using A)

$$\frac{x - 1}{\frac{1}{\sqrt{3}}} = \frac{y - 2}{\frac{1}{\sqrt{3}}} = \frac{z + 3}{-\frac{1}{\sqrt{3}}}$$

or $$\frac{x - 1}{1} = \frac{y - 2}{1} = \frac{z + 3}{-1}$$

or $$x - 1 = y - 2 = -(z + 3)$$

Example 35.2 Find the equations of a line passing through the point $(1, -3, 2)$ and having direction ratios $(1, -2, 3)$

Solution : The equations of the line are (Using B)

$$\frac{x - 1}{1} = \frac{y + 3}{-2} = \frac{z - 2}{3}$$

Example 35.3 Find the equations of the line passing through two points $(1, -3, 2)$ and $(4, 2, -3)$

Solution : The equations of the required line are (Using C)

$$\frac{x-1}{4-1}=\frac{y+3}{2+3}=\frac{z-2}{-3-2} \quad \text{or} \quad \frac{x-1}{3}=\frac{y+3}{5}=\frac{z-2}{-5}$$

Example 35.4 Find the equations of the line passing through the points $(1,-5,-6)$ and parallel to the line joining the points $(0,2,3)$ and $(-1,3,7)$.

Solution : Direction ratios of the line joining the points $(0,2,3)$ and $(-1,3,7)$ are

$$-1-0,\ 3-2,\ 7-3$$

or $$-1,+1,+4$$

$\therefore$ Direction ratios of a line parallel to this line can be taken as $-1,1,4$.

Thus, equations of the line through the point $(1,-5,-6)$ and parallel to the given line are

$$\frac{x-1}{-1}=\frac{y+5}{1}=\frac{z+6}{4}$$

CHECK YOUR PROGRESS 35.1

1. Find the equations, in symmetric form, of the line passing through the point $(1,-2,3)$ with direction ratios $3,-4,5$.
2. Find the equations of the line, in symmetric form, passing through the points $(3,-9,4)$ and $(-9,5,-4)$.
3. Find the equations of the line, in symmetric form, passing through the points $(-7,5,3)$ and $(2,6,8)$
4. Find the equations of the line, in symmetric form, through the point (1,2,3) and parallel to the line joining the points $(-4,7,2)$ and $(5,-3,-2)$
5. Find the equations of a line passing through the origin and equally inclined to the co-ordinate axes.

35.2 REDUCTION OF THE EQUATIONS OF A LINE INTO SYMMETRIC FORM

You may recall that a line can be thought of as the intersection of two non-parallel planes. Let the equations of the two intersecting planes be

$$ax+by+cz+d=0 \qquad \text{...(i)}$$

and $$a'x+b'y+c'z+d'=0 \qquad \text{....(ii)}$$

Let AB be the line of intersection of the two planes. Every point on the line AB lies on both the planes. Thus, the co-ordinates of any point on the line satisfy the two equations of the planes. Hence (i) and (ii) together represent the equations of a line.

The equations $ax + by + cz = 0$ and $a'x + b'y + c'z = 0$ together represent the equations of the line through the origin parallel to the above line as the above two planes also pass through origin. The above form of the equations of a line is referred to as general (***or non-symmetric***) form of the equations of a line.

To reduce the general equations of a line given by (i) and (ii) in the symmetric form, we need the direction cosines of the line as well as the co-ordinates of a point on the line.

Let the direction cosines of the line be l, m and n. The line is perpendicular to the normal to planes given by (i) and (ii).

$$\therefore \quad al + bm + cn = 0 \quad \text{and} \quad a'l + b'm + c'n = 0$$

By cross multiplication method, we get

$$\frac{l}{bc' - b'c} = \frac{m}{ca' - ac'} = \frac{n}{ab' - a'b}$$

Thus, the direction cosines of the line are proportional to

$$(bc' - b'c), (ca' - ac') \text{ and } (ab' - a'b).$$

The point where the line meets the XY – plane is obtained by putting $z = 0$ in the equations (i) and (ii), which give

$$ax + by + d = 0 \qquad \text{....(iii)}$$

$$a'x + b'y + d' = 0 \qquad \text{....(iv)}$$

Solving (iii) and (iv), we get

$$x = \frac{bd' - b'd}{ab' - a'b}, \; y = \frac{da' - d'a}{ab' - a'b}$$

$$\therefore \text{ A point on the line is } \left(\frac{bd' - b'd}{ab' - a'b}, \frac{da' - d'a}{ab' - a'b}, 0 \right)$$

∴ The equations of the line in symmetric form are

$$\frac{x - \dfrac{bd' - b'd}{ab' - a'b}}{bc' - b'c} = \frac{y - \dfrac{da' - d'a}{ab' - a'b}}{ca' - c'a} = \frac{z}{ab' - a'b}$$

Note : Instead of taking $z = 0$, we may take $x = 0$ or $y = o$ or any other suitable value for any of the x, y, z provided the two equations so obtained have a unique solution.

Example 35.5 Convert the equations of the line given by $x - 2y + 3z = 4$, $2x - 3y + 4z = 5$ into symmetric form and find its direction cosines.

Solution : Let $z = 0$ be the z – co-ordinate of a point on each of the planes.

∴ The equations of the planes reduce to

$$x - 2y = 4$$

$$2x - 3y = 5$$

which on solving give $x = -2$ and $y = -3$

$\therefore$ The point common to two planes is $(-2, -3, 0)$.

Let l, m, n be the direction cosines of the line As the line is perpendicular to normal to the planes. we have

$$l - 2m + 3n = 0$$

and $$2l - 3m + 4n = 0$$

$\therefore$ $$\frac{l}{-8+9} = \frac{m}{6-4} = \frac{n}{-3+4}$$

or $$\frac{l}{1} = \frac{m}{2} = \frac{n}{1} = \pm\frac{1}{\sqrt{6}}$$

$\therefore$ The equations of the line are

$$\frac{x+2}{\pm\frac{1}{\sqrt{6}}} = \frac{y+3}{\pm\frac{2}{\sqrt{6}}} = \frac{z}{\pm\frac{1}{\sqrt{6}}}$$

or $$\frac{x+2}{1} = \frac{y+3}{2} = \frac{z}{1}$$

and the direction cosines of the line are $\pm\frac{1}{\sqrt{6}}, \pm\frac{2}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}$ (the same sign positive or negative to be taken throughout)

CHECK YOUR PROGRESS 35.2

1. Find the equations, in symmetric form, of the line given by

   (i) $x + 5y - z = 7$ and $2x - 5y + 3z = -1$

   (ii) $x + y + z + 1 = 0$ and $4x + y - 2z + 2 = 0$

   (iii) $x - y + z + 5 = 0$ and $x - 2y - z + 2 = 0$

35.3 PERPENDICULAR DISTANCE OF A POINT FROM A LINE

Let P be the point (x_1, y_1, z_1) and AQ be the given line whose equations are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

where l, m and n are the direction cosines of the line AQ, Q is the foot of the perpendicular from P on AQ and A is the point (α, β, γ).

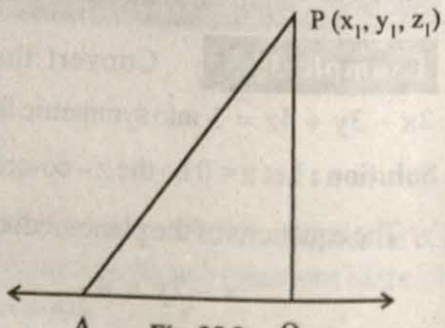

Fig. 35.3

We have $$PQ^2 = AP^2 - AQ^2$$

Now $$AP^2 = (x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2$$

Again AQ, the projection of AP on the line is

$$(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n$$

$$\therefore \quad PQ^2 = \left\{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2\right\} - \left\{(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n\right\}^2$$

which gives the length of perpendicular (PQ) from the point P to the line.

Example 35.6 Find the distance of a point (2,3,1) from the line

$$y + z - 1 = 0 = 2x - 3y - 2z + 4$$

Solution : Let z = 0 be the z-coordinate of the point common to two planes.

$\therefore$ Their equations become $y = 1$ and $2x - 3y + 4 = 0$ which give $x = -\frac{1}{2}$

$\therefore$ A point common to two planes is $\left(-\frac{1}{2}, 1, 0\right)$

Let l, m, n be the direction cosines of the given line

Then, $0l + m + n = 0$ and $2l - 3m - 2n = 0$

or $$\frac{l}{1} = \frac{m}{2} = \frac{n}{-2} = \frac{1}{\pm 3} \quad \text{or} \quad l = \pm\frac{1}{3},\ m = \pm\frac{2}{3},\ n = \mp\frac{2}{3}$$

If PQ is the length of the perpendicular from (2,3,1) to the given line. Then

$$PQ^2 = \left[\left(2+\frac{1}{2}\right)^2 + (3-1)^2 + (1-0)^2\right] - \left[\frac{5}{2}\times\frac{1}{3} + \frac{2}{3}\times 2 - 1\times\frac{2}{3}\right]^2$$

$$= \left(\frac{25}{4} + 4 + 1\right) - \left(\frac{5}{6} + \frac{4}{3} - \frac{2}{3}\right)^2$$

$$= \frac{45}{4} - \frac{9}{4} = 9$$

$\therefore$ PQ = 3

Thus, the required distance is 3 units.

CHECK YOUR PROGRESS 35.3

1. Find the distance of the point from the line, for each of the following :

(i) Point (0, 2, 3), line $\frac{x+3}{3} = \frac{y-1}{2} = \frac{z+4}{3}$

(ii) Point $(-1, 3, 9)$, line $\frac{x-13}{5} = \frac{y+8}{-6} = \frac{z-31}{1}$

(iii) Point (4, 1, 1), line $x + y + z = 4,\ x - 2y - z = 4$

(iv) Point (3, 2, 1), line $x + y + z = 4,\ x - 2y - z = 4$

Notes

35.4 ANGLE BETWEEN A LINE AND A PLANE

The angle between a line and a plane is the complement of the angle between the line and normal to the plane. Let the equations of the line be

$$\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n} \qquad .....(i)$$

and that of the plane be

$$ax + by + cz + d = 0 \qquad ....(ii)$$

If θ be the angle between (i) and (ii), then

$$\sin\theta = \cos(90^\circ - \theta) = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\ \sqrt{a^2 + b^2 + c^2}}$$

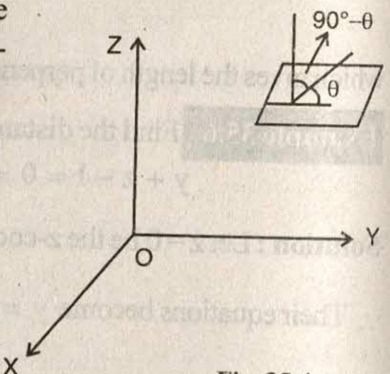

Fig. 35.4

Example 35.7 Find the angle between the line $\frac{x-2}{3} = \frac{y+3}{3} = \frac{z-1}{1}$ and the plane $2x - 3y + 4z - 7 = 0$

Solution : Here the angle θ between the given line and given plane is given by

$$\sin\theta = \frac{2\times 3 - 3\times 3 + 4\times 1}{\sqrt{3^2 + 3^2 + 1^2}\ \sqrt{2^2 + (-3)^2 + 4^2}} = \frac{1}{\sqrt{19}\ \sqrt{29}}$$

$$= \frac{1}{\sqrt{551}}$$

or $$\theta = \sin^{-1}\left(\frac{1}{\sqrt{551}}\right)$$

CHECK YOUR PROGRESS 35.4

1. Find the angle between the following lines and the planes.

(i) Line : $\frac{x-4}{1} = \frac{y+2}{4} = \frac{z-3}{-1}$ and Plane : $3x - 4y + 5z = 5$

(ii) Line : $\frac{x-2}{2} = \frac{z-3}{3} = \frac{y+2}{1}$ and Plane : $-2x + 4y - 5z = 20$

(iii) Line : $\frac{x}{4} = \frac{y-2}{-3} = \frac{y+2}{5}$ and Plane : $x - 4y + 6z = 11$

(iv) Line : $\frac{x+2}{4} = \frac{y-3}{5} = \frac{z+4}{1}$ and Plane : $4x - 3y - z - 7 = 0$

Notes

35.5 CONDITION OF COPLANARITY OF TWO LINES

If the two lines given by

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad \text{......(i)}$$

and

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad \text{.....(ii)}$$

intersect, they lie in the same plane.

Equation of a plane containing line (i) is

$$A(x-x_1) + B(y-y_1) + C(z-z_1) = 0 \quad \text{......(iii)}$$

with $$Al_1 + Bm_1 + Cn_1 = 0 \quad \text{.....(iv)}$$

If the plane (iii) contains line (ii), the point (x_2, y_2, z_2) must lie on it.

Thus, $$A(x_2-x_1) + B(y_2-y_1) + C(z_2-z_1) = 0 \quad \text{.....(v)}$$

with $$Al_2 + Bm_2 + Cn_2 = 0 \quad \text{....(vi)}$$

Eliminating A,B and C from (iv), (v) and (vi), we have

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \quad \text{.....(vii)}$$

which is the necessary condition for coplanarity of lines given by (i) and (ii)

Again, eliminating A,B and C from (iii), (iv) and (vi) we get

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \quad \text{.....(viii)}$$

(viii) represents the equation of the plane containing the two intersecting lines.

We shall now show that if the condition (vii) holds, then the lines (i) and (ii) are coplanar.

Consider the plane

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad \text{....(ix)}$$

or, $$(x - x_1)(m_1n_2 - m_2n_1) + (y - y_1)(n_1l_2 - n_2l_1) + (z - z_1)(l_1m_2 - l_2m_1) = 0$$

A line will lie in the plane, if the normal to the plane is perpendicular to the line and any point on the line lies in the plane.

You may see that

$$l_1(m_1n_2 - m_2n_1) + m_1(n_1l_2 - n_2l_1) + n_1(l_1m_2 - l_2m_1) = 0$$

Hence line (i) lies in plane (ix)

By similar argument, we can say that line (ii) lies on plane (ix)

∴ The two lines are coplanar.

Thus, the condition (vii) is also sufficient for the two lines to be coplanar.

Corollary : The lines (i) and (ii) will intersect if and only if (vii) holds and lines are not parallel.

Note :

(i) Two lines in space, which are neither intersecting nor parallel, do not lie in the same plane. Such lines are called skew lines.

(ii) If the equation of one line be in symmetric form and the other in general form, we proceed as follows:

Let equations of one line be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad \text{.... (i)}$$

and that of the other line be

$$ax + by + cz + d = 0 \text{ and } a'x + b'y + c'z + d' = 0 \qquad \text{......(ii)}$$

If the two lines are coplanar, then a point on the first line should satisfy equations of the second line. A general point on line (i) is $(x_1 + lr, y_1 + mr, z_1 + nr)$.

This point lies on $ax + by + cz + d = 0$ if

$$a(x_1 + lr) + b(y_1 + mr) + c(z_1 + nr) + d = 0$$

or $$r = -\frac{ax_1 + by_1 + cz_1 + d}{al + bm + cn}$$

Similarly, this point should lie on $a'x + b'y + c'z + d' = 0$, resulting in

$$r = -\frac{a'x_1 + b'y_1 + c'z_1 + d'}{a'l + b'm + c'n}$$

Equating the two values of r obtained above, we have the required condition as

$$\frac{ax_1 + by_1 + cz_1 + d}{al + bm + cn} = \frac{a'x_1 + b'y_1 + c'z_1 + d'}{a'l + b'm + c'n}$$

> *Note :* In case, both the lines are in general form, convert one of them into symmetric form and then proceed as above.

Example 35.8 Prove that the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$

and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ are co-planar.

Solution : For the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ ......(i)

and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ .... (ii)

to be coplanar we must have

$$\begin{vmatrix} 8-5 & 4-7 & 5+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$$

or $\quad 3(12+5) + 3(12+35) + 8(4-28) = 0$

or $\quad 51 + 141 - 192 = 0$

or $\quad 0 = 0$ which is true.

∴ The two lines given by (i) and (ii) are coplanar.

Example 35.9 Prove that the lines

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} \text{ and } \frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7}$$

are coplanar. Find the equation of the plane containing these lines.

Solution : For the lines

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} \text{ and } \frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7}$$

to be coplanar, we must have

$$\begin{vmatrix} 2+1 & 4+3 & 6+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3 & 7 & 11 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

or $$3(35-28)-7(21-7)+11(12-5)=0$$

or $$21-98+77=0$$

or $$0=0.$$ which is true.

$\therefore$ The given lines are coplanar.

Equation of the plane containing these lines is

$$\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

or $$(x+1)(35-28)-(y+3)(21-7)+(z+5)(12-5)=0$$

or $$7x+7-14y-42+7z+35=0$$

or $$7x-14y+7z=0$$

or $$x-2y+z=0$$

CHECK YOUR PROGRESS 35.5

1. Prove that the following lines are coplanar :

   (i) $\frac{x-3}{3}=\frac{y-2}{-4}=\frac{z+1}{1}$ and $x+2y+3z=0=2x+4y+3z+3$

   (ii) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $4x-3y+1=0=5x-3z+2$

2. Show that the lines $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$

   and $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ are coplanar. Find the equation of the plane containing them.

LET US SUM UP

- A line is the intersection of two non-parallel planes.
- The equations of a line in symmetric form are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

where (x_1, y_1, z_1) is a point on the line and l, m and n are its direction cosines (or direction ratios).

Notes

- The general form of the equations of a line is

$$a_1x + b_1y + c_1z + d_1 = 0$$

$$a_2x + b_2y + c_2z + d_2 = 0$$

- Equations of a line in two-point form are $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) are two points on the line.

- The angle θ between the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane $ax + by + cz + d = 0$ is given by

$$\sin\theta = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\,\sqrt{a^2 + b^2 + c^2}}$$

- The condition of coplanarity of two lines,

$$\frac{x - x_1}{l} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \text{ and } \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

is
$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing the lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

SUPPORTIVE WEB SITES

http:// www.wikipedia.org.
http:// mathworld.wolfram.com

TERMINAL EXERCISE

1. Find the equations of the line passing through the points (1,4,7) and (3, −2, 5)
2. Find the equations of the line passing through the point (−1, −2, −3) and perpendicular to the plane $3x - 4y + 5z - 11 = 0$

3. Find the direction cosines of the line which is perpendicular to the lines whose direction ratios are $1, -1, 2$ and $2, 1, -1$.

4. Show that the line segment joining the points (1,2,3) and (4,5,7) is parallel to the line segment joining the points $(-4, 3, -6)$ and $(2, 9, 2)$

5. Find the angle between the lines

$$\frac{x-1}{2} = \frac{y-2}{-4} = \frac{z+5}{5} \quad \text{and} \quad \frac{x+1}{3} = \frac{y+1}{4} = \frac{z}{2}$$

6. Find the equations of the line passing through the point $(1, 2, -4)$ and perpendicular to each of the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \quad \text{and} \quad \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$$

7. Convert the equations of the line $x - y + 2z - 5 = 0$, $3x + y + z - 6 = 0$ into the symmetric form.

8. Show that the lines $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z}{-1}$ and $\frac{x-4}{3} = \frac{y-1}{-2} = \frac{z-1}{1}$ are coplanar. Find the equation of the plane containing them.

9. Find the equation of the plane containing the lines.

$$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5} \quad \text{and} \quad \frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$$

10. Find the projection of the line segment joining the points (2,3,1) and (5,8,7) on the line $\frac{x}{2} = \frac{y+4}{3} = \frac{z+1}{6}$

Notes

ANSWERS

CHECK YOUR PROGRESS 35.1

1. $\frac{x-1}{3}=\frac{y+2}{-4}=\frac{z-3}{5}$ 2. $\frac{x-3}{-6}=\frac{y+9}{7}=\frac{z-4}{-4}$

3. $\frac{x+7}{9}=\frac{y-5}{1}=\frac{z-3}{5}$ 4. $\frac{x-1}{9}=\frac{y-2}{-10}=\frac{z-3}{-4}$

5. $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$

CHECK YOUR PROGRESS 35.2

1. (i) $\frac{x-2}{2}=\frac{y-1}{-1}=\frac{z}{-3}$ (ii) $\frac{x+\frac{1}{3}}{1}=\frac{y+\frac{2}{3}}{-2}=\frac{z}{1}$

   (iii) $\frac{x-1}{3}=\frac{y-3}{2}=\frac{z+3}{1}$

CHECK YOUR PROGRESS 35.3

1. (i) $\sqrt{21}$ units (ii) 21 units

   (iii) $\sqrt{\frac{27}{14}}$ units (iv) $\sqrt{6}$ units

CHECK YOUR PROGRESS 35.4

1. (i) $\sin^{-1}\left(-\frac{3}{5}\right)$ (ii) $\sin^{-1}\left(\frac{1}{\sqrt{70}}\right)$

   (iii) $\sin^{-1}\left(\frac{46}{\sqrt{2650}}\right)$ (iv) 0°

CHECK YOUR PROGRESS 35.5

2. $x+y+z=0$

TERMINAL EXERCISE

1. $\frac{x-1}{2}=\frac{y-4}{-6}=\frac{z-7}{-2}$ 2. $\frac{x+1}{3}=\frac{y+2}{-4}=\frac{z+3}{5}$

3. $-\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}$

5. 90°

6. $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$

7. $\frac{x-\frac{11}{4}}{-3} = \frac{y+\frac{9}{4}}{5} = \frac{z}{4}$

8. $2x - 5y - 16z + 13 = 0$

9. $17x - 47y - 24z + 172 = 0$

10. $\frac{57}{7}$ units.

36

THE SPHERE

You must have played or seen students playing football, basketball or table tennis. Football, basketball, table tennis ball are all examples of geometrical figures which we call "spheres" in three dimensional geometry.

If we consider the rotation of a semi-circle OAPB about its diameter AB, the rotation generates a sphere whose centre is the centre of the semi-circle and whose radius is equal to the radius of the semi-circle.

Thus, a sphere is the locus of a point in space which moves in such a way that its distance from a fixed point, in space, always remains constant. The fixed point is called the **centre** of the sphere and the fixed distance is called the **radius** of the sphere.

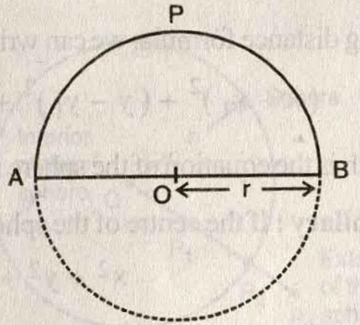

Fig. 36.1

The difference between a sphere and a circle is that a sphere is a figure in three-dimensional space while a circle is a figure in two dimensions (in a plane).

In this lesson, we shall study the equation of a sphere in centre-radius form, equation of a sphere through four non-coplanar points, the equation of a sphere in diameter form, plane section of a sphere and general equation of a sphere through a given circle.

OBJECTIVES

After studying this lesson, you will be able to find:

- the equation of a sphere in centre-radius form;
- the equation of a sphere in general form;
- the equation of a sphere through four non-coplanar points;
- the equation of a sphere in diameter form;
- the equation of a plane section of a sphere; and
- the general equation of a sphere through a given circle.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of two-dimensional coordinate geometry.
- Knowledge of three-dimensional geometry.
- Various forms of the equations of a plane.
- Straight line in space.

36.1 EQUATION OF A SPHERE IN CENTRE RADIUS FORM

Recall that a sphere is the set of points equidistant from a fixed point. The fixed point is called the **centre** of the sphere and the constant (or fixed) **distance** is its radius.

Let P (x, y, z) be a point on the sphere whose centre is $C(x_1, y_1, z_1)$. Let r be the radius of the sphere.

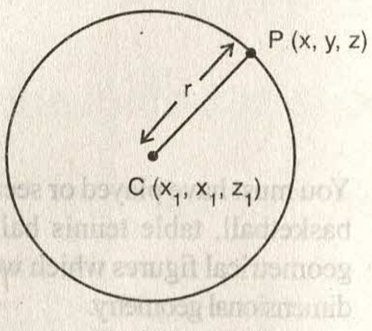

Fig. 36.2

$$\therefore \qquad CP^2 = r^2 \qquad \text{.....(i)}$$

Using distance formula, we can write (i) as

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2 \text{ .....(A)}$$

which is the equation of the sphere in **centre-radius form.**

Corollary : If the centre of the sphere is at the origin, the equation of sphere with radius r is

$$x^2 + y^2 + z^2 = r^2 \qquad \text{..........(ii)}$$

Note : The equation (A) is an equation of second degree in x, y and z.

We observe that :

(a) The co-efficients of the terms involving x^2, y^2 and z^2 are all equal (in this case each is equal to 1).

(b) There are no terms involving the products xy, yz or zx.

Thus, you will see that a general equation of second degree in x,yand z will represent a sphere if it satisfies the above two conditions.

(c) Consider an equation of the form

$$ax^2 + ay^2 + az^2 + 2lx + 2my + 2nz + d = 0 \qquad (a \neq 0) \qquad \text{.....(iii)}$$

On dividing throughtout by 'a', equation (iii) can be written as

$$x^2 + y^2 + z^2 + \frac{2l}{a}x + \frac{2m}{a}y + \frac{2n}{a}z + \frac{d}{a} = 0 \qquad \text{.....(B)}$$

(B) can be written in the form $x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0$ .....(iv)

where $g = \frac{l}{a}$, $f = \frac{m}{a}$, $h = \frac{n}{a}$ and $c = \frac{d}{a}$.

Equation (iv) can be written as

$$(x+g)^2+(y+f)^2+(z+h)^2=\left(g^2+f^2+h^2-c\right) \quad .....(v)$$

Comparing (v) with (A) above, we have centre of the sphere (iv) as $(-g,-f,-h)$ and radius of the sphere (iv) as $\sqrt{g^2+f^2+h^2-c}$

Equation (iv) is called the general form of the equation of the sphere.

In order that the sphere may be real $g^2+f^2+h^2-c \geq 0$.

(d) In case r = 0, the sphere is a point sphere.

(e) The sphere whose centre is same as the centre of (iv) is called a concentric to sphere in (iv).The equation of the sphere concentric with (iv) is $x^2+y^2+z^2+2gx+2fy+2hz+k=0$ where k is a constant and this can determined from some other condition.

36.1.1 Interior and Exteroir of Sphere

Let O be the centre of a sphere with radius r. A point P_1 lies in the interior of the sphere if $OP_1 < r$. The point P_2 lies on the sphere if $OP_2 = r$ and a point P_3 lies in the exterior of the sphere if $OP_3 > r$.

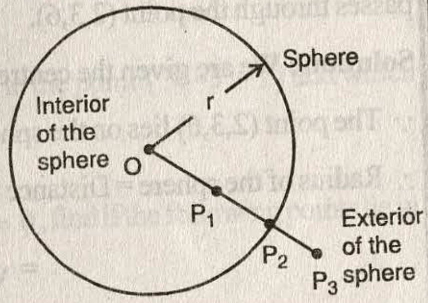

Fig. 36.3

Example 36.1 Find the equation of the sphere with centre at origin and radius 4.

Solution : The required equation of the sphere is

$$(x-0)^2+(y-0)^2+(z-0)^2=4^2$$

or $$x^2+y^2+z^2=16$$

Example 36.2 Find the equation of the sphere with centre at $(2,-3,1)$ and radius $\sqrt{7}$.

Solution : The required equation of the sphere is

$$(x-2)^2+[y-(-3)]^2+(z-1)^2=\left(\sqrt{7}\right)^2$$

or $$(x-2)^2+(y+3)^2+(z-1)^2=7$$

or $$x^2-4x+4+y^2+6y+9+z^2-2z+1-7=0$$

or $$x^2+y^2+z^2-4x+6y-2z+7=0$$

Example 36.3 Find the centre and radius of the sphere whose equation is

$$2x^2+2y^2+2z^2-4x+8y-6z-19=0$$

Solution : The given equation of the sphere is

$$2x^2 + 2y^2 + 2z^2 - 4x + 8y - 6z - 19 = 0$$

or $$x^2 + y^2 + z^2 - 2x + 4y - 3z - \frac{19}{2} = 0$$

∴ Here $$g = -1, f = 2, h = -\frac{3}{2}, c = -\frac{19}{2}$$

∴ Centre $$= (-g, -f, -h) = \left(1, -2, \frac{3}{2}\right)$$

and radius $$= \sqrt{g^2 + f^2 + h^2 - c} = \sqrt{(-1)^2 + (2)^2 + \left(-\frac{3}{2}\right)^2 + \frac{19}{2}}$$

$$= \sqrt{1 + 4 + \frac{9}{4} + \frac{19}{2}} = \frac{\sqrt{67}}{2}$$

Example 36.4 Find the equation of the sphere which has its centre at the origin and which passes through the point (2,3,6).

Solution : We are given the centre of the sphere as (0, 0, 0).

∵ The point (2,3,6) lies on the sphere.

∴ Radius of the sphere = Distance of the origin from the point (2,3,6)

$$= \sqrt{(2-0)^2 + (3-0)^2 + (6-0)^2}$$

$$= \sqrt{4 + 9 + 36}$$

$$= \sqrt{49}$$

$$= 7$$

∴ Equation of the required sphere is

$$(x-0)^2 + (y-0)^2 + (z-0)^2 = 7^2$$

or $$x^2 + y^2 + z^2 = 49$$

Example 36.5 For the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z - 2 = 0$, find if the point (2,3,4) lies in the interior or exterior of the sphere.

Solution : The equation of given sphere is

$$x^2 + y^2 + z^2 - 2x + 4y - 6z - 2 = 0.$$

Here $$g = -1, f = 2, h = -3 \text{ and } c = -2$$

∴ Centre of the sphere = $(1, -2, 3)$

Radius of the sphere = $\sqrt{1 + 4 + 9 + 2} = 4$

The distance of the point (2,3,4) from centre

$$= \sqrt{(2-1)^2 + (3+2)^2 + (4-3)^2} = 3\sqrt{3}\,.$$

As $3\sqrt{3} > 4$, the point (2,3,4) lies in the exterior of the given sphere.

CHECK YOUR PROGRESS 36.1

1. Find the equation of the sphere whose centre is at the origin and whose radius is 5.
2. Find the centre and radius of the sphere

$$3x^2 + 3y^2 + 3z^2 - 3x + 6y - 9z - 17 = 0$$

3. Find the equation of the sphere which passes through the origin and

   (i) has the centre at the point $(3,-3,-1)$.

   (ii) has the centre at the point $(2,-2,-1)$.

4. Find the equation of the sphere which has its centre at the point $(3,-3-1)$ and which passes through the point $(5,-2,1)$
5. For the sphere $x^2 + y^2 + z^2 - 6x + 8y - 2z + 1 = 0$, find if the following points lie in the exterior, interior or on the sphere

   (i) $(2,-3,4)$ (ii) $(-1,-4,-2)$ (iii) $(-1,2,3)$

36.2 EQUATIONS OF A SPHERE THROUGH FOUR NON-COPLANAR POINTS

Recall that the general equation of a sphere is

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0 \qquad .....(i)$$

This contains four constants g, f, h and c. If somehow we are able to determine the values of these constants, we can determine the equation of the sphere.

If it is given that the sphere passes through four non-coplanar points, it will give us four equations which will enable us to evaluate the four constants.

Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) be four non-coplanar points. These points will satisfy equation (i) as they lie on it.

$$\therefore \quad x_1^2 + y_1^2 + z_1^2 + 2gx_1 + 2fy_1 + 2hz_1 + c = 0 \qquad .....(1)$$

$$x_2^2 + y_2^2 + z_2^2 + 2gx_2 + 2fy_2 + 2hz_2 + c = 0 \qquad .....(2)$$

$$x_3^2 + y_3^2 + z_3^2 + 2gx_3 + 2fy_3 + 2hz_3 + c = 0 \qquad .....(3)$$

$$x_4^2 + y_4^2 + z_4^2 + 2gx_4 + 2fy_4 + 2hz_4 + c = 0 \qquad .....(4)$$

Solving (1), (2), (3) and (4) for g, f, h and c and substituting in (i), we get the required equation of sphere as

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_2^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Note : The student is not expected to evaluate the above determinant.

Example 36.6 Find the equation of the sphere passing through the points (0, 0, 0) (1, 0, 0), (0, 1, 0) and (0, 0, 1). Find also its radius.

Solution : Let the equation of the sphere be

$$x^2+y^2+z^2+2gx+2fy+2hz+c=0 \quad .....(i)$$

Since it passes through the point (0, 0, 0)

$\therefore \quad c = 0$

Again (i) pass through the point (1, 0, 0)

$\therefore \quad 1+0+0+2g+0+0+0=0 \text{ or } g=-\frac{1}{2} \quad [\text{As } c=0]$

Similarly, since it passes through the points (0, 1, 0) and (0, 0, 1)

we have, $f=-\frac{1}{2}$ and $h=-\frac{1}{2}$

$\therefore$ The equation (i) reduces to $x^2+y^2+z^2-x-y-z=0$

which is the required equation of the sphere.

Radius of the sphere $=\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-0}=\frac{\sqrt{3}}{2}$.

Example 36.7 Find the equation of the sphere which passes through the origin and the points $(2,1,-1)$, $(1,5,-4)$ and $(-2,4,-6)$. Find its centre and radius.

Solution : Let the equation of the sphere be

$$x^2+y^2+z^2+2gx+2fy+2hz+c=0 \quad .....(i)$$

Since it passes through the point (0, 0, 0), c = 0

Again (i) passes through the point (2, 1, – 1)

$\therefore \quad 2^2+1^2+(-1)^2+4g+2f-2h=0$

or $\quad 6 + 4g + 2f - 2h = 0$

or $\quad 3 + 2g + f - h = 0 \quad$ .....(ii)

Similarly, as it passes through the points (1, 5, – 4) and (–2, 4, – 6)

We have have $42 + 2g + 10f - 8h = 0 \quad$ .....(iii)

$$56 - 4g + 8f - 12h = 0 \quad \text{.....(iv)}$$

Substituting the value of h from (ii) in (iii) and (iv) we get,

$$42 + 2g + 10f - 8(3 + 2g + f) = 0$$

or $\quad 18 - 14g + 2f = 0$

or $\quad 9 - 7g - f = 0 \quad$ .....(v)

and $\quad 56 - 4g + 8f - 12(3 + 2g + f) = 0$

or $\quad 20 - 28g - 4f = 0$

or $\quad 5 - 7g - f = 0 \quad$ .....(vi)

Solving (v) and (vi) for g and f, we get $g = 1, f = -2$.

Putting $g = 1$ and $f = -2$ in (ii), we get $h = 3$.

$\therefore$ The required equation of the sphere is

$$x^2 + y^2 + z^2 + 2x - 4y + 6z = 0$$

Centre of the sphere is $(-1, 2, -3)$.

and radius $= \sqrt{(1)^2 + (-2)^2 + 3^2 - 0} = \sqrt{14}$

Example 36.8 Find the equation of the sphere which passes through the points $(2, 3, 0)$, $(3, 0, 2)$ $(0, 1, 3)$ and $(2, 2, 0)$.

Solution : The required equation of the sphere is

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ 2^2 + 3^2 + 0^2 & 2 & 3 & 0 & 1 \\ 3^2 + 0^2 + 2^2 & 3 & 0 & 2 & 1 \\ 0^2 + 1^2 + 3^2 & 0 & 1 & 3 & 1 \\ 2^2 + 2^2 + 0^2 & 2 & 2 & 0 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ 13 & 2 & 3 & 0 & 1 \\ 13 & 3 & 0 & 2 & 1 \\ 10 & 0 & 1 & 3 & 1 \\ 8 & 2 & 2 & 0 & 1 \end{vmatrix} = 0$$

Solving (1), (2), (3) and (4) for g, f, h and c and substituting in (i), we get the required equation of sphere as

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ {x_1}^2+{y_1}^2+{z_1}^2 & x_1 & y_1 & z_1 & 1 \\ {x_2}^2+{y_2}^2+{z_2}^2 & x_2 & y_2 & z_2 & 1 \\ {x_3}^2+{y_2}^2+{z_3}^2 & x_3 & y_3 & z_3 & 1 \\ {x_4}^2+{y_4}^2+{z_4}^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Note : The student is not expected to evaluate the above determinant.

Example 36.6 Find the equation of the sphere passing through the points (0, 0, 0) (1, 0, 0), (0, 1, 0) and (0, 0, 1). Find also its radius.

Solution : Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0 \quad .....(i)$$

Since it passes through the point (0, 0, 0)

$\therefore \quad c = 0$

Again (i) pass through the point (1, 0, 0)

$$\therefore \quad 1 + 0 + 0 + 2g + 0 + 0 + 0 = 0 \text{ or } g = -\frac{1}{2} \quad [\text{As } c = 0]$$

Similarly, since it passes through the points (0, 1, 0) and (0, 0, 1)

we have, $f = -\frac{1}{2}$ and $h = -\frac{1}{2}$

$\therefore$ The equation (i) reduces to $x^2 + y^2 + z^2 - x - y - z = 0$

which is the required equation of the sphere.

Radius of the sphere $= \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 0} = \frac{\sqrt{3}}{2}$.

Example 36.7 Find the equation of the sphere which passes through the origin and the points $(2, 1, -1)$, $(1, 5, -4)$ and $(-2, 4, -6)$. Find its centre and radius.

Solution : Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0 \quad .....(i)$$

Since it passes through the point (0, 0, 0), c = 0

Again (i) passes through the point (2, 1, – 1)

$$\therefore \quad 2^2 + 1^2 + (-1)^2 + 4g + 2f - 2h = 0$$

or $\qquad 6 + 4g + 2f - 2h = 0$

or $\qquad 3 + 2g + f - h = 0 \qquad$ .....(ii)

Similarly, as it passes through the points $(1, 5, -4)$ and $(-2, 4, -6)$

We have have $42 + 2g + 10f - 8h = 0 \qquad$ .....(iii)

$$56 - 4g + 8f - 12h = 0 \qquad \text{.....(iv)}$$

Substituting the value of h from (ii) in (iii) and (iv) we get,

$$42 + 2g + 10f - 8(3 + 2g + f) = 0$$

or $\qquad 18 - 14g + 2f = 0$

or $\qquad 9 - 7g - f = 0 \qquad$ .....(v)

and $\qquad 56 - 4g + 8f - 12(3 + 2g + f) = 0$

or $\qquad 20 - 28g - 4f = 0$

or $\qquad 5 - 7g - f = 0 \qquad$ .....(vi)

Solving (v) and (vi) for g and f, we get $g = 1$, $f = -2$.

Putting $g = 1$ and $f = -2$ in (ii), we get $h = 3$.

$\therefore$ The required equation of the sphere is

$$x^2 + y^2 + z^2 + 2x - 4y + 6z = 0$$

Centre of the sphere is $(-1, 2, -3)$.

and radius $= \sqrt{(1)^2 + (-2)^2 + 3^2 - 0} = \sqrt{14}$

Example 36.8 Find the equation of the sphere which passes through the points $(2, 3, 0)$, $(3, 0, 2)$ $(0, 1, 3)$ and $(2, 2, 0)$.

Solution : The required equation of the sphere is

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ 2^2 + 3^2 + 0^2 & 2 & 3 & 0 & 1 \\ 3^2 + 0^2 + 2^2 & 3 & 0 & 2 & 1 \\ 0^2 + 1^2 + 3^2 & 0 & 1 & 3 & 1 \\ 2^2 + 2^2 + 0^2 & 2 & 2 & 0 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ 13 & 2 & 3 & 0 & 1 \\ 13 & 3 & 0 & 2 & 1 \\ 10 & 0 & 1 & 3 & 1 \\ 8 & 2 & 2 & 0 & 1 \end{vmatrix} = 0$$

CHECK YOUR PROGRESS 36.2

1. Find the equation of the sphere which passes through the four non-coplanar points given below :

   (i) (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c)

   (ii) (0, 0, 0), (–a, b, c), (a, – b, c) and (a, b, – c)

   (iii) (0, 0, 0), (0, 2, – 1), (–1, 1, 0) and (1, 2, – 3)

   Find the centre and radius of each of the above spheres obtained.

2. Find the equation of the sphere passing through the points $(1,-1,-1), (3, 3, 1), (-2, 0, 5)$ and $(-1, 4, 4)$.

36.3 DIAMETER FORM OF THE EQUATION OF A SPHERE

Let the point $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the extremities of a diameter of a sphere with centre O. Let $P(x, y, z)$ be any point on the sphere.

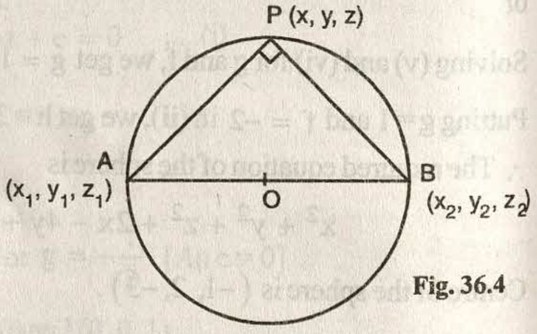

Fig. 36.4

∴ PA and PB are at right angles to each other.

The direction ratios of PA and PB are $(x - x_1)$, $(y - y_1)$, $(z - z_1)$ and $(x - x_2)$, $(y - y_2)$, $(z - z_2)$ respectively.

Since PA and PB are at right angles.

$$\therefore (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0 \quad \text{....(i)}$$

which is the equation of the sphere in ***diameter form***

Note : The equation (i) can be rewritten as

$$x^2 + y^2 + z^2 - (x_1 + x_2)x - (y_1 + y_2)y - (z_1 + z_2)z$$
$$+ x_1x_2 + y_1y_2 + z_1z_2 = 0 \quad \text{....(A)}$$

Let us try to find the equation of the sphere by an ***alternative method***

As O is the mid-point of AB

∴ The co-ordinates of O are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$

and radius of the sphere is

$$\frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

∴ The equation of sphere is

$$\left[x - \frac{(x_1 + x_2)}{2}\right]^2 + \left[y - \frac{(y_1 + y_2)}{2}\right]^2 + \left[z - \frac{(z_1 + z_2)}{2}\right]^2$$

$$=\frac{1}{4}(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$$

$$=\left(\frac{x_2-x_1}{2}\right)^2+\left(\frac{y_2-y_1}{2}\right)^2+\left(\frac{z_2-z_1}{2}\right)^2$$

which on simplification gives

$$x^2+y^2+z^2-(x_1+x_2)x-(y_1+y_2)y-(z_1+z_2)z$$

$$+\left[\left(\frac{x_2+x_1}{2}\right)^2-\left(\frac{x_2-x_1}{2}\right)^2+\left(\frac{y_2+y_1}{2}\right)^2-\left(\frac{y_2-y_1}{2}\right)^2\right.$$

$$\left.+\left(\frac{z_2+z_1}{2}\right)^2-\left(\frac{z_2-z_1}{2}\right)^2\right]=0$$

or $$x^2+y^2+z^2-(x_1+x_2)x-(y_1+y_2)y-(z_1+z_2)z$$

$$+\frac{1}{4}(4x_1x_2+4y_1y_2+4z_1z_2)=0$$

or $$x^2+y^2+z^2-(x_1+x_2)x-(y_1+y_2)y-(z_1+z_2)z$$

$$+(x_1x_2+y_1y_2+z_1z_2)=0$$

which is the same as (A) above.

Example 36.9 Find the equation of the sphere having extremities of one of its diameter as the points (2,3, 5) and (–4, 7, 11). Find its centre and radius also.

Solution : The required equation of the sphere is

$$(x-2)(x+4)+(y-3)(y-7)+(z-5)(z-11)=0$$

or $$x^2+y^2+z^2+2x-10y-16z+68=0$$

Centre of the sphere is $(-1, 5, 8)$ and radius

$$=\sqrt{(-1)^2+5^2+8^2-68}=\sqrt{22}.$$

Example 36.10 One end of a diameter of the sphere $x^2+y^2+z^2-2x-6y-2z+2=0$ is the point $(3, 4, -1)$. Find the other end of the diameter.

Solution : Let the other end of the diameter be the point (x_1, y_1, z_1).

∴ The equation of the sphere described on the join of two given points (x_1, y_1, z_1) and $(3, 4, -1)$ is

$$(x-x_1)(x-3)+(y-y_1)(y-4)+(z-z_1)(z+1)=0$$

or $$x^2+y^2+z^2-(x_1+3)x-(y_1+4)y-(z_1-1)z+3x_1+4y_1-z_1=0 \qquad \text{.....(i)}$$

Now equation (i) is identical with the given sphere

$$x^2+y^2+z^2-2x-6y-2z+2=0 \qquad \text{.....(ii)}$$

Comparing the co-efficients of each terms of the equation (ii)

We have, $x_1+3=2$ or $x_1=-1$

$y_1+4=6$ or $y_1=2$

$z_1-1=2$ or $z_1=3$

and $3x_1+4y_1-z_1=2$ .....(iii)

$x_1=-1$, $y_1=2$ and $z_1=3$ satisfy (iii) also.

∴ The co-ordinates of the other end of the diameter are $(-1, 2, 3)$.

Example 36.11 Find the centre and radius of the sphere

$$(x-2)(x-4)+(y-1)(y-3)+(z-2)(z+3)=0$$

Solution : The equation of the sphere is

$$(x-2)(x-4)+(y-1)(y-3)+(z-2)(z+3)=0$$

or $$x^2-6x+8+y^2-4y+3+z^2+z-6=0$$

or $$x^2+y^2+z^2-6x-4y+z+5=0$$

Here, $g=-3$, $f=-2$ and $h=\frac{1}{2}$.

∴ Centre of the sphere is $\left(3, 2, -\frac{1}{2}\right)$ and radius is

$$\sqrt{(3)^2+(2)^2+\left(-\frac{1}{2}\right)^2-5}$$

or $$\sqrt{9+4+\frac{1}{4}-5} \quad \text{or} \quad \frac{\sqrt{33}}{2}$$

CHECK YOUR PROGRESS 36.3

1. Find the equation of a sphere whose extremities of one of the diameter are

   (i) $(2,-3,4)$ and $(-5, 6,-7)$

   (ii) $(2,-3,4)$ and $(-1,0,5)$

(iii) $(5,4,-1)$ and $(-1,2,3)$

Also find the centre and radius of each of the above spheres.

2. If one end of a diameter of the sphere $x^2+y^2+z^2-2x+4y-6z-7=0$ be the point $(-1,2,4)$, find the other end of the diameter.

3. Find the centre and radius of the sphere

$$(x+1)(x+2)+(y-3)(y-5)+(z-7)(z+3)=0$$

36.4 PLANE SECTION OF A SPHERE AND A SPHERE THROUGH A GIVEN CIRCLE

(i) Let us first consider the case of plane section of a sphere.

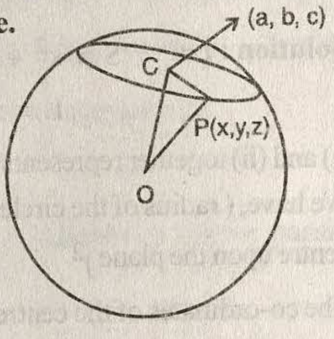

Fig. 36.5

We know that the equation of the sphere with origin as centre and radius r is given by $x^2+y^2+z^2=r^2$ .....(i)

Let C (a, b, c) be the centre of the plane section of the sphere whose equation we have to find out. The line segment OC drawn from O to the plane through C (a,b,c) is normal to the plane. The direction ratios of the normal OC are a, b, c.

Let P(x,y,z) be any point on the plane section.

The direction ratio of PC are $x-a,\ y-b,\ z-c$ and it is perpendicular to OC.

Using the condition of perpendicularity, we have

$$(x-a)a+(y-b)b+(z-c)c=0 \qquad \text{....(ii)}$$

Equation (ii) is satisfied by the co-ordinates of any point P on the plane. Hence (i) and (ii) together constitute the equation of the plane section of the sphere.

Corollary 1 : If the centre of the sphere is $(-g,-f,-h)$, then the direction ratios of the normal OC are $a+g,\ b+f,\ c+h$.

Hence $(x-a)(a+g)+(y-b)(b+f)+(z-c)(c+h)=0$ .....(A)

and the equation of sphere is $x^2+y^2+z^2+2gx+2fy+2hz+d=0$ .....(B)

(A) and (B) together, in this case, constitute the equation of plane section of the sphere.

Corollary 2 : The plane section of a sphere by a plane passing through the centre of the sphere is called a **Great Circle**. You can see that

(i) the centre of this section coincides with the centre of the sphere. You may note that a great circle is a circle with greatest radius amongest all the plane section of the sphere.

(ii) the radius of the great circle is equal to the radius of sphere.

Now we consider the case of the sphere through a given circle.

(1) Let us consider the sphere

$$S \equiv x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0 \qquad .....(i)$$

and the plane $L \equiv lx + my + nz + k = 0$ .....(ii)

Equations (i) and (ii) together represent the equation of a circle, being the intersection of $S = 0$ and $L = 0$.

The equation $S + \lambda L = 0$, ...(iii)

where λ is a constant, gives the equation of the sphere passing through the circle given by (i) and (ii) together, for $S + \lambda L = 0$ is satisfied by the co-ordinates of the points lying on the circle.

Example 36.12 Find the centre and radius of the circle given by the equations

$$x^2 + y^2 + z^2 - 6x - 4y + 12z - 36 = 0, \; x + 2y - 2z = 1$$

Solution : Let $S \equiv x^2 + y^2 + z^2 - 6x - 4y + 12z - 36 = 0$ .....(i)

$L \equiv x + 2y - 2z - 1 = 0$ .....(ii)

(i) and (ii) together represents the equation of a circle.

We have, $(\text{radius of the circle})^2 = (\text{radius of the sphere})^2 - (\text{perpendicular distance from the centre upon the plane})^2$

The co-ordinates of the centre of (i) are $(3, 2, -6)$

$$\therefore \text{ Radius} = \sqrt{3^2 + 2^2 + (-6)^2 + 36} = \sqrt{85}$$

Perpendicular distance from the centre $(3, 2, -6)$ upon the plane (ii)

$$= \frac{1.3 + 2.2 - 2(-6) - 1}{\sqrt{1^2 + 2^2 + 2^2}} = 6$$

$\therefore$ $(\text{Radius of the circle})^2 = 85 - 36 = 49$ or, radius $= 7$

The equation

$$x^2 + y^2 + z^2 - 6x - 4y + 12z - 36 + \lambda(x + 2y - 2z - 1) = 0$$

represents a sphere passing through (i) and (ii)

The above equation can be rewritten as

$$x^2 + y^2 + z^2 - (6 - \lambda)x - (4 - 2\lambda)y + (12 - 2\lambda)z - 36 - \lambda = 0 \qquad ....(iii)$$

Now, the centre of the sphere given in (iii) is $\left(\frac{6-\lambda}{2}, 2-\lambda, \lambda - 6\right)$

and radius is $\sqrt{\left(\frac{6-\lambda}{2}\right)^2 + (2-\lambda)^2 + (\lambda-6)^2 + 36 + \lambda}$

$$\therefore \quad \left(\frac{6-\lambda}{2}\right)^2 + (2-\lambda)^2 + (\lambda-6)^2 + 36 + \lambda = 7^2$$

or $\quad 36 - 12\lambda + \lambda^2 + 4(4 - 4\lambda + \lambda^2) + 4(\lambda^2 - 12\lambda + 36) + 4(36 + \lambda) = 196$

or $\quad 9\lambda^2 - 72\lambda + 144 = 0$

or $\quad \lambda^2 - 8\lambda + 16 = 0$

or $\quad (\lambda - 4)^2 = 0$

or $\quad \lambda = 4, 4$

$\therefore$ The co-ordinates of the centre are $\left(\frac{6-4}{2}, 2-4, 4-6\right)$ i.e., $(1, -2, -2)$

Hence, the required radius and centre of the circle are 7 and $(1, -2, -2)$ respectively.

Example 36.13 Find the equation of the sphere for which the circle given by

$x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$ and $2x + 3y + 4z - 8 = 0$ is a great circle.

Solution : The equation

$x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda(2x + 3y + 4z - 8) = 0$ represents a sphere passing through the circle given by

$$x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$$

and $$2x + 3y + 4z - 8 = 0$$

The centre of this sphere is $\left(-\lambda, -\frac{(7 + 3\lambda)}{2}, 1 - 2\lambda\right)$.

This should satisfy $2x + 3y + 4z - 8 = 0$, as its centre should coincide with the centre of the sphere

$\therefore \quad -2\lambda - \frac{21 + 9\lambda}{2} + 4 - 8\lambda = 8$

or, $\quad -4\lambda - 21 - 9\lambda + 8 - 16\lambda = 16 \quad$ or $\quad \lambda = -1$

$\therefore$ The required equation of the sphere is

$$x^2 + y^2 + z^2 + 7y - 2z + 2 - 2x - 3y - 4z + 8 = 0$$

or, $$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$$

CHECK YOUR PROGRESS 36.4

1. Find the centre and radius of each of the following circles :

   (i) $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$ and $x + 2y + 2z = 15$

   (ii) $(x - 3)^2 + (y + 2)^2 + (z - 1)^2 = 100$ and $2x - 2y - z + 9 = 0$

2. Show that the circle in which the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z = 2$ is cut by the plane $x + 2y + 2z = 20$ has its centre at the point (2, 4, 5) with a radius of $\sqrt{3}$ units.

LET US SUM UP

- A sphere is the set of points in space such that its distance from a fixed point always remains constant. The fixed point in the space is called the centre and the constant distance is called the radius of the sphere.
- The equation of a sphere with centre (x_1, y_1, z_1) and radius r is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

This is called ***centre radius*** form of the equation of a sphere.

- The equation of a sphere with centre at origin and radius r is

$$x^2 + y^2 + z^2 = r^2$$

- A general equation of second degree in x, y and z represents a sphere if

(i) The co-efficients of the terms involving x^2, y^2 and z^2 are all equal.

(ii) There are no terms involving xy, yz or zx.

- The centre and radius of a sphere

$x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0$ are $(-g, -f, -h)$

and $\sqrt{g^2 + f^2 + h^2 - c}$ respectively.

- The equation of a sphere through four non-coplanar points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ and (x_4, y_4, z_4) is

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

- The equation of a sphere, with $(x_1, y_1, z_1), (x_2, y_2, z_2)$ as extremities of a diameter, is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$$

- The plane section of a sphere with origin as centre and r, the radius is determined by joint equations :

$$x^2 + y^2 + z^2 = r^2 \text{ and } a(x - a) + b(y - b) + c(z - c) = 0$$

- The sphere $S \equiv x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + c = 0$ and the plane $L \equiv lx + my + nz + k = 0$, together represent the equation of a circle being intersection of S = 0 and L = 0.

  The equation $S + \lambda L = 0$ represents the equation of the sphere passing through the circle given by S = 0 and L = 0 together.

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. Find the equation of the sphere which passes through the point $(2, -2, 3)$ and which has its centre at (0, 0, 0).
2. Find the co-ordinates of the centre and the radius of the sphere

$$2\left(x^2 + y^2 + z^2\right) - 4x + 6y - 5z = 0$$

3. Find the equation of the sphere concentric with

   $x^2 + y^2 + z^2 - 2x - 4y - 6z - 11 = 0$ and with radius 10.
4. Find the locus of a point which moves such that the sum of the squares of its distances from the points (1, 2, 3), (2, –3, 5) and (0, 7, 4) is 147.
5. For the sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z + 5 = 0$, find if the point $(1, -1, 3)$ lies in the interior or exterior of the sphere.
6. Find the equation of the sphere passing through the points (0, 0, 0), (1, 0, 0), (0, 2, 0) and (0, 0,3).
7. Find the equation of the sphere having extremities of one of its diameter as $(-1, 2, -3)$ and $(3, 1, -1)$.
8. If one end of a diameter of the sphere $x^2 + y^2 + z^2 - 7x - 3y + 1 = 0$ is the point (4, 5, 1), find the other end point of the diameter.
9. Find the equation of the sphere passing through the origin and cutting intercepts a, b and c from the positive directions of the co-ordinate axes.
10. Find the radius of the circular section of the sphere $x^2 + y^2 + z^2 = 49$ by the plane $2x + 3y - z - 5\sqrt{14} = 0$.
11. Find the equation of a sphere for which the circle $x^2 + y^2 + z^2 = 4, x + y + 4 = 0$ is a great circle.

ANSWERS

CHECK YOUR PROGRESS 36.1

Notes

1. $x^2 + y^2 + z^2 = 25$

2. Centre : $\left(\frac{1}{2}, -1, \frac{3}{2}\right)$; radius $= \sqrt{\frac{55}{6}}$

3. (i) $(x-3)^2 + (y+3)^2 + (z+1)^2 = 19$

   (ii) $(x-2)^2 + (y+2)^2 + (z+1)^2 = 9$

4. $(x-3)^2 + (y+3)^2 + (z+1)^2 = 9$

5. (i) interior (ii) on the sphere (iii) exterior

CHECK YOUR PROGRESS 36.2

1. (i) $x^2 + y^2 + z^2 - ax - by - cz = 0, \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right); \frac{\sqrt{a^2+b^2+c^2}}{2}$

   (ii) $\frac{x^2+y^2+z^2}{a^2+b^2+c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0;$

   $$\left(\frac{a^2+b^2+c^2}{2a}, \frac{a^2+b^2+c^2}{2b}; \frac{a^2+b^2+c^2}{2c}\right);$$

   $$\frac{a^2+b^2+c^2}{2\,abc}\sqrt{b^2c^2+c^2a^2+a^2b^2}$$

   (iii) $6\left(x^2+y^2+z^2\right) + 14x + 2y + 34z = 0; \left(-\frac{7}{6}, -\frac{1}{6}, \frac{17}{6}\right); \frac{\sqrt{339}}{6}$

2.
$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 3 & 1 & -1 & -1 & 1 \\ 19 & 3 & 3 & 1 & 1 \\ 29 & -2 & 0 & 5 & 1 \\ 33 & -1 & 4 & 4 & 1 \end{vmatrix} = 0$$

CHECK YOUR PROGRESS 36.3

1. (i) $x^2 + y^2 + z^2 + 3x - 3y + 3z - 56 = 0; \left(-\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right); \frac{\sqrt{251}}{2}$

(ii) $x^2 + y^2 + z^2 - x + 3y - 9z + 18 = 0$; $\left(\frac{1}{2}, -\frac{3}{2}, \frac{9}{2}\right)$; $\frac{\sqrt{19}}{2}$

(iii) $x^2 + y^2 + z^2 - 4x - 6y - 2z = 0$; $(2, 3, 1)$; $\sqrt{14}$

2. $(3, -6, 2)$ 3. $\left(-\frac{3}{2}, 4, 2\right)$; $\frac{\sqrt{105}}{2}$

CHECK YOUR PROGRESS 36.4

1. (i) $(1, 3, 4)$; $\sqrt{7}$ (ii) $(-1, 2, 3)$; 8

TERMINAL EXERCISE

1. $x^2 + y^2 + z^2 = 17$ 2. $\left(1, -\frac{3}{2}, \frac{5}{4}\right)$; $\frac{\sqrt{77}}{4}$

3. $x^2 + y^2 + z^2 - 2x - 4y - 6z - 86 = 0$

4. $x^2 + y^2 + z^2 - 2x - 4y - 8z - 10 = 0$

5. Interior of the sphere

6. $x^2 + y^2 + z^2 - x - 2y - 3z = 0$

7. $x^2 + y^2 + z^2 - 2x - 3y + 4z + 2 = 0$

8. $(3, -2, -1)$ 9. $x^2 + y^2 + z^2 - ax - by - cz = 0$

10. $2\sqrt{6}$ 11. $x^2 + y^2 + z^2 + 4x + 4y + 12 = 0$.

SENIOR SECONDARY COURSE
MATHEMATICS

Questions For Practice - Vectors and Three Dimensional Geometry

---

Maximum Marks : 50 **Time : $1\frac{1}{2}$ Hours**

Instructions :

1. Answer all questions on a separate sheet of paper.
2. Give the following information on your answer sheet :
   - Name
   - Enrolment number
   - Subject
   - Assignment number
   - Address

- Get your assignment checked by the subject teacher at your study centre so that you get positive feedback about your performance.

Do not send your assignment to the National Institute of Open Schooling

1. The position vectors of two points A and B are $2\vec{a}+3\vec{b}$ and $3\vec{a}+\vec{b}$ respectively. Find $\overrightarrow{AB}$. (2)

2. Given $\vec{a}=\hat{i}-\hat{j}-\hat{k}$ and $\vec{b}=2\hat{i}-4\hat{j}+3\hat{k}$, find the magnitude of $\vec{a}-\vec{b}$. (2)

3. If $\vec{a}=3\hat{i}+2\hat{j}-6\hat{k}$ and $\vec{b}=4\hat{i}-5\hat{j}+\hat{k}$, find $\vec{a}\cdot\vec{b}$ (2)

4. Show that the points (4, 7, 8), (2, 3, 4), (–1, –2, 1) and (1, 2, 5) are the vertices of a parallelogram. (2)

5. Show that the points P, Q and R with position vector $\vec{a}-2\vec{b}$, $2\vec{a}+3\vec{b}$ and $-7\vec{b}$ respectively are collinear. (3)

6. Show that the vectors $\vec{a}-2\vec{b}, 3\vec{a}+\vec{b}$ and $\vec{a}+4\vec{b}$ are coplanar. (3)

7. Find a unit vector in the direction of $2\vec{a}+3\vec{b}$ where $\vec{a}=\hat{i}+3\hat{j}+\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}-\hat{k}$. (3)

8. Find the equation of the plane which divides the line joining the points (2, 1, 4) and (2, 4, 6) internally in the ratio of 2 : 3. (3)

9. If following forces (expressed in Newtons) act on a particle P :

   $\vec{F}_1=2\hat{i}+\hat{j}-3\hat{k}$, $\vec{F}_2=-3\hat{i}+2\hat{j}+2\hat{k}$, $\vec{F}_3=3\hat{i}-2\hat{j}+\hat{k}$

   Find (i) the resultant of the forces (ii) the magnitude of the resultant (4)

10. Determine the value of k for which the planes

$3x - 2y + kz - 1 = 0$ and $x + ky + 5z + 2 = 0$ are perpendicular to each other. (4)

11. Find the equation of the line passing through the point $(-1, -2, -3)$ and perpendicular to the plane $3x - 4y + 5z - 11 = 0$ (4)

12. Find the equation of the planes represented by

$$6x^2 - 12y^2 + 4z^2 + xy + 13yz - 14xz = 0$$

Also find the angle between the planes. (6)

13. Find the projection of the line segment joining the points (2, 3, 1) and (5, 8, 7) on the line

$$\frac{x}{2} = \frac{y+4}{3} = \frac{z+1}{6}$$ (6)

14. Find the centre and radius of the following circle :

$$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0 \text{ and } x + 2y + 2z - 15 = 0$$ (6)

(OPTIONAL -II)

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A Word With You

Dear Learner,

This module contains five lessons pertaining to your day to day activities. You must have noticed advertisement in the newspaper for selling shares of various companies. Have you ever thought what are shares and debentures, how selling of shares helps a company to accumulate funds? How are they bought or sold and why people like investing in them? You will get some of the answers in the lesson on **Shares and Debentures.**

You must have known that people insure household items against theft, fire and natural calamities. We also insure ourselves against accident, riot etc. The lesson on **Insurance** will introduce you to different types of policies, their objectives, how to calculate premium or surrender value when a policy is in force. You will also learn about Mediclaim insurance in this lesson.

The Government has a social responsibility towards the nation and has also to look after the security and development of the country. This cannot be done without resources, which have to be mobilized primarily through the process of taxation both direct and indirect. These will be discussed in the lesson on **Indirect Taxes.**

Application of Mathematics in Industry, Trade, Commerce, Economics, Insurance and Taxation will also be discussed in this module.

We would suggest that you go through all the solved examples given in the learning material and then try to solve independently all questions included in "**Check Your Progress**" and "**Terminal Exercise**" given at the end of each lesson.

This module contains a "**Questions For Practice**" page for self-check. Try to solve it and get it checked by the subject teacher at your study centre.

If you face any difficulty please do write to us. Your suggestions are also welcome.

Good luck!

Yours

(Suvendu Sekhar Das)

Academic Officer (Mathematics)

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OPTIONAL-II
MATHEMATICS FOR COMMERCE, ECONOMICS AND BUSINESS

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37

SHARES AND DEBENTURES

To start an industry on a large scale, requires huge amount of capital, professional skills, and other resources. Sometime it may not be possible for a single individual to do the needful. In such cases, a group of like minded people get together and set up a company, called a Joint-Stock Company registered under companies act. The people who start the company are called promoters of the company, who frame the constitution of the company, which lays down the objectives of the company.

To raise the capital from the general public, the company issues a prospectus giving details of the projects undertaken, background of the company, its strength and risks involved. The capital of the company is divided into convenient units of equal value, called shares. Normally, they are of the denomination of Rs. 10 or Rs.100.

In this lesson, we shall study about shares, stocks and method of calculating dividends/profits by the share holders. We shall also study about debentures, and the method of sale and purchase of shares and debentures.

OBJECTIVES

After studying this lesson, you will be able to

- explain the need of a joint stock company;
- define the terms such as shares, preferred and common shares, debentures, dividend, brokerage, paid-up value, at par, at premium, below par, etc.;
- distinguish between shares and debentures;
- find the dividend under diferent conditions for different types of shares;
- find the change in income when different transactions are made;
- distinguish between various stocks on the basis of income therefrom; and
- find the income (as interest) on a given number of debentures.

EXPECTED BACKGROUND KNOWLEDGE

- knowledge of unitary method, ratio and proportion, simple interest

37.1 SOME DEFINITIONS

Share : The total capital of the company is divided into convenient units of equal value and each unit is called a share.

Shareholder : An individual who purchases/possesses the share/shares of the company is called a shareholder of the company. Each share holder is issued a share certificate by the company, indicating the number of shares purchased and value of each share.

Par value : The original value of the share, which is written on the share certificate, is called its par value. This is also called nominal value or face value of the share.

Dividend : When the company starts production and starts earning profit, after retaining some profit for running expenses interest on loans, if raised, the remaining part of the profit is divided among shareholders, and is called dividend. Dividend is usually expressed as certain percentage of its par value or certain among per share.

37.2 TYPES OF SHARES

The shares are of two types :

(i) Preferred shares, and

(ii) Ordinary (or common) shares

We will discuss in detail about them below:

(i) **Preferred shares :** There are the shares on which some fixed amount of dividend is paid, after working expenses taxes, interests, etc. are paid, Sometimes, when the profit is not enough even to meet the other expenses, even the preferred share holders do not get any dividend.

(ii) **Ordinary (or common) shares :** These type of shareholders get dividend only after the holders of preference shares receive their share of profit. Due to this only the rate of dividend is not fixed and keep on varying.

37.3 FACE VALUE AND MARKET VALUE OF A SHARE

We have already explained that the price at which the company issues shares to the shareholders is called its face value.

Like other commodities, shares are also sold and purchased in the market. This market is given a special name "Stock Exchange ". The price of a share as quoted in the market, is called the market value of the share. Like other commodities, the market value of shares keep on chainging according to demand in the market.

(i) When the market value of a share equals its face value, the share is said to be at par.

(ii) If the market value of a share is more than its face value, it is said tobe above par (or at premium). On the contrary, if the market price of a share is less than its face value, it is said to be below par (or at discount)

Notes

Note : The company always pays dividend on the face value and not on the market value of the share.

Let us take some examples

Example 37.1 Ram Lal has 200 shares of par value Rs. 10 each. The company declares an annual dividend of 8%. Find the dividend received by Ram Lal.

Solution : Total par value of 200 shares owned by Ram Lal

$$= \text{Rs. } (200\times10) = \text{Rs.}2000$$

$$\therefore \quad \text{Dividend} = \text{Rs. } (2000\times8\%)$$

$$= \text{Rs. } 160$$

Example 37.2 A company has issued 25000 shares of par value Rs. 100 each. If the total dividend declared by the company is Rs.200000, find the rate of dividend paid by the company.

Solution : Number of shares : 25000

Total dividend paid = Rs. 200000

$$\therefore \quad \text{Dividend paid per share} = \text{Rs. } \frac{200000}{25000} = \text{Rs. } 8$$

Par value of a share = Rs. 100

$$\therefore \quad \text{Rate of dividend paid} = 8\ \%$$

Example 37.3 Ram had 2000 preferred shares and 5000 ordinary shares of a company of par value Rs. 10 each. If the dividend declared on preferred shares is 20% and is 12% on ordinary shares, find the annual dividend received by Ram.

Solution : Dividend on preferred shares = 20% of total par value

$$= \text{Rs. } \left(\frac{20}{100} \times 2000 \times 10\right) = \text{Rs. } 4000$$

Dividend on ordinary shares = 12% of total par value

$$= \text{Rs. } \left(\frac{12}{100} \times 5000 \times 10\right) = \text{Rs. } 6000$$

$$\therefore \text{ Total dividend received by Ram} = \text{Rs. } (4000+6000) = \text{Rs. } 10000$$

Example 37.4 Lalwani purchases 1000 shares of a company, of par value Rs.100 each, paying an annual dividend of 20%, at such a price that he gets $12\frac{1}{2}$% on his investment. Find the market price of a share

Solution : Dividend received by Lalwani

$$= \text{Rs. } \left(\frac{1000 \times 100 \times 20}{100}\right) = \text{Rs.}20000$$

Let the investment of Lalwani be Rs. x

$$\therefore \quad 12\tfrac{1}{2}\%\text{ of } x = \text{Rs. } 20000$$

$$\therefore \quad x = \text{Rs.}\left(\frac{20000 \times 100 \times 2}{25}\right) = \text{Rs.}160000$$

$\therefore$ Market value of 1000 shares = Rs. 160000

$\therefore$ Market value of one share = Rs. 160

Notes

Example 37.5 A man sells 1000 shares of a company (of par value Rs. 10 each) giving a dividend of 20% at Rs.25 each. He reinvests the proceeds in shares of another company (of par value Rs. 100 each), paying a dividend of $12\frac{1}{2}$%, at Rs. 125 each. Find the change in his income.

Solution : Dividend From 1000 shares of par value Rs. 10 each

$$= \text{Rs.}\left(\frac{1000 \times 10 \times 20}{100}\right) = \text{Rs.}2000$$

Amount received by selling 1000 shares at Rs. 25 each

$$= \text{Rs. } (1000 \times 25) = \text{Rs. } 25000$$

$\therefore$ Number of shares of other company at market rate of Rs. 125 each $= \dfrac{25000}{125} = 200$

Par value of each share = Rs. 100 : Rate of dividend : $12\frac{1}{2}$%

$$\therefore \quad \text{Dividend received} = \left(\frac{200 \times 100 \times 25}{200}\right) = \text{Rs.}2500$$

$\therefore$ Change in income = Rs. (2500-2000)

= Rs.500 increase.

Example 37.6 Asif purchased 400 shares of a company (of par value Rs. 10 each) at a premium of 25%. He sells these shares when their price rose to Rs. 16.50 per share. Find his gain in the transaction.

Solution : Purchase price of a share by Asif $= \text{Rs.}\left[10 \times \left(\frac{100 + 25}{100}\right)\right]$

$= \text{Rs.}12\frac{1}{2}$

Selling price of each share $= \text{Rs. } 16\frac{1}{2}$

$\therefore$ Gain on selling one share $= \text{Rs.}\left(16\frac{1}{2} - 12\frac{1}{2}\right) = \text{Rs.}4$

$\therefore$ Gain on selling 400 shares = Rs. 1600

CHECK YOUR PROGRESS 37.1

1. A company declares an annual dividend of 9%. If Ramesh owns 600 shares of the company, of par value Rs.10 each, find the dividend received by Ramesh.

2. Find the dividend received annually on 500 shares of par value Rs. 100 each at 5% payable half yearly.
3. A company issued 25000 shares of par value Rs. 10 each. If the total dividend released by the company is Rs. 40000, find the rate of dividend paid by the company.
4. The capital stock of company is Rs. 2750000, which is divided into 2500 preferred shares of 7% of par value Rs.100 and 25000 ordinary shares of par value Rs.10 each. If the net profit shown by the company is Rs. 270000 out of which Rs. 160000 is distributed as dividend, find

   (i) rate of dividend paid on ordinary shares

   (ii) dividend to be received by a person for 100 preferred shares and 200 ordinary shares.
5. A man buys 400 shares of a company (of par value Rs.100 each) for Rs.125 per share and sells them at a premium of Rs. 45 per share. Find the gain of the man in the transaction.
6. Raman purchased 250 shares of a company (of par value Rs.100 each, paying an annual dividend of 20%) at such a price that he gets 10% on his investment. Find the market price of a share.
7. Ravi sells 200 shares of a company (of par value Rs.100 each and paying a dividend 7.5%) at Rs.120 each. He reinvests the proceeds in another company in shares of par value Rs.10 at Rs. 12.50, paying a dividend of 10%. Find the change in Ravi's dividend income.
8. Akash sells 2000 shares of a company A (of par value Rs.100 each, paying a dividend of 10%) at Rs.120 each. He invested the proceeds in shares of another company B par value Rs. 25 each at Rs. 40 giving dividend of 15%

   Find (i) the number of shares of company B purchased by Akash

   (ii) change in the dividend income.

37.4 STOCKS AND BROKERAGE

37.4.1 Stock

If a person holds 500 shares of a company , of par value Rs.100 each, he is said to hold Rs. (500×100) or Rs. 50000 stock of that company.

Generally, the stocks are referred to by their rates of dividend. Thus, if the dividend on Rs.100 stock is Rs.10, it is called "10% stock" and if the market value of the stock is Rs.125, we say it is "10% stock at Rs.125". It may be recalled that dividend is always calculated on par value.

37.4.2 Brokerage

The sale and purchase of a stock in market is generally done through a broker, who charges some amount for his services. This charge is called brokerage.

The brokerage is generally charged as some percentage of the market value of the stock or it is some fixed amount on each unit of stock.

37.4.3 Rule for brokerage

(i) the brokerage is added to the market value of the stock, when it is purchased

(ii) The brokerage is subtracted from the market value when it is sold

37.5 INCOME ON A STOCK

Let us illustrate the method of calculating income on a stock through examples.

Examples 37.7 Find the income on 8 % stock of Rs. 46000 purchased at Rs.120

Solution : Here, face value of the stock = Rs. 46000

$$\text{Income on Rs. } 100 = \text{Rs. } 8$$

$$\therefore \quad \text{Income on Rs. } 46000 = \text{Rs}\left(\frac{8}{100} \times 46000\right) = \text{Rs. } 3680$$

Example 37.8 A man invested Rs. 36300 at Rs. 120 in 12% stock. Find his income if brokerage is Re.1

Solution : According to rule (i) above, market value

$$= \text{Rs.}(120+1) = \text{Rs.}121$$

$$\therefore \quad \text{Amount of stocks bought} = \text{Rs. } \frac{36300 \times 100}{121} = \text{Rs. } 30000$$

$$\therefore \quad \text{Income} = \text{Rs. } \frac{30000 \times 12}{100} = \text{Rs. } 3600$$

Example 37.9 What amount of 9 % stock will produce an annual income of Rs.16200 after paying an income tax of 10% on the dividend ?

Solution : Income on Rs. 100 stock = Rs. 9

$$\text{Income tax} = \text{Rs. } \left(\frac{9 \times 10}{100}\right) = \text{Rs.}0.90$$

$$\therefore \quad \text{Net income on Rs.}100 = \text{Rs.}(9.0 - 0.9) = \text{Rs. } 8.10$$

If the income is Rs. 8.10, then investment = Rs. 100

$$\text{If income is Rs. 16200, then investment} = \text{Rs. } \left(\frac{1000 \times 16200}{81}\right)$$

$$= \text{Rs. } 200000$$

37.6 MARKET VALUE OF A STOCK

If the face value of a stock is given its market value can be found by finding the market value of each unit of stock. Let us illustrate is through examples.

Examples 37.10 How much money should be invested to purchase Rs. 9600 stock at 95 ?

Solution : Market value of Rs. 100 stock = Rs. 95

$$\therefore \quad \text{Market value of Rs. 9600 stock} = \text{Rs. } \left(\frac{95 \times 9600}{100}\right) = \text{Rs. } 9120$$

Example 37.11 A man sells Rs. 4250 of 8 % stock at 96 and invests the proceeds in 12% stock at 102. How much stock does he hold now ?

Solution : A stock of Rs. 100 is sold for Rs. 96

$\therefore$ A stock of Rs. 4250 is sold for Rs. $\left(\dfrac{96 \times 4250}{100}\right) =$ Rs. 4080

In the second case

An investment of Rs. 102 gets a stock of Rs. 100

$\therefore$ Investment of Rs.4080 gets a stock of Rs. $\left(\dfrac{100 \times 4080}{102}\right) =$ Rs. 4000

37.7 SALE AND PURCHASE OF STOCKS

The market value of stocks go on changing according to market conditions. Some stockholders sell their stocks and reinvest the proceeds in a more beneficial stocks giving more income. Let us see this through examples.

Example 37.12 A man bought Rs. 12000 of 10% stock at 92 and sold it when the price rose to Rs. 98. Find his total gain and gain percent.

Solution : Investment in buying Rs. 12000 stock = Rs. $\left(\dfrac{92 \times 12000}{100}\right) =$ Rs.11040

Amount received by selling the stock = Rs. $\left(\dfrac{98 \times 12000}{100}\right) =$ Rs. 11760

Gain = Rs. (11760-11040) = Rs.720

$\therefore$ Gain percent $= \dfrac{720}{11040} \times 100 = 6.52$

Example 37.13 A man invests Rs. 28000 in 7 % stock at 98 and sells it when the price rose to Rs. 105. He reinvests the sale proceeds in 12% stock at 120. Find the change in his income.

Solution : Income from first stock = Rs. $\left(\dfrac{7 \times 28000}{98}\right) =$ Rs. 2000

Amount realised from selling the stock = Rs. $\left(\dfrac{105 \times 28000}{98}\right) =$ Rs. 30000

Income from second stock = Rs. $\left(\dfrac{30000 \times 12}{120}\right) =$ Rs. 3000

$\therefore$ Increase in income = Rs. (3000-2000) = Rs.1000

Example 37.14 Ram Lal sells 450 shares of Rs. 10 of a company at Rs. 18 each, which pays a dividend of 9% and then invests the proceeds of this sale in the purchase of Rs.5 shares of another company at Rs.4.50 each, giving a dividend of $3\frac{1}{2}$%. Find the change in his income

Solution : Dividend on 450 shares = Rs. $\left(\frac{450 \times 10 \times 9}{100}\right)$ = Rs. 405

Selling price of 450 shares = Rs. (450×18) = Rs.8100

Number of shares of second company bought = $\left(\frac{8100 \times 2}{9}\right) = 1800$

∴ Income from second company = Rs. $\left(\frac{1800 \times 5 \times 7}{100 \times 2}\right)$ = Rs. 315

∴ There is a decrease of Rs.(405−315) = Rs.90 in his income

37.8 DIFFERENT STOCKS - PART INVESTMENT

For maximising income, an investor can invest his capital in a number of different stocks. His total income will be the sum of the incomes from different stocks.Conversely, if the respective incomes are given, the investment in different stocks can also be determined.

Example 37.15 Mohan invested a part of Rs.15000 in 5 % stock at 90 and the remaining in 7% stock at 120. If his total income from the stocks is Rs. 855, find the respective investments in different stocks.

Solution :

(i) Let the investment in 5 % stock = Rs. x

(ii) The investment in 7% stock = Rs (15000−x)

∴ Income from (i) stock = Rs. $\frac{x \times 5}{90}$ = Rs. $\frac{x}{18}$

Income from (ii) stock = Rs. $(15000 - x)\frac{7}{120}$

∴ Total income $= 855 = \frac{x}{18} + \frac{105000 - 7x}{120}$

or $855 \times 360 = 20x + 105000 \times 3 - 21x$

∴ $x = 315000 - 307800$

$= 7200$

∴ Amount invested in 5 % stock = Rs. 7200

Amount invested in 7 % stock = Rs. (15000 − 7200) = Rs. 7800

Example 37.16 A invested Rs. 188000, partly in 6 % stock at 88 and the remaining in 5 % stock at 99. If the incomes derived from both the stocks are the same, find the respective investments in two stocks.

Solution : Let the investment in 6 % stock at 88 be Rs. x

∴ The investment in 5 % stock at 99 = Rs. (188000 − x)

$$\text{Income from first stock} = \text{Rs.}\left(\frac{x \times 6}{88}\right) \quad \text{.... (i)}$$

$$\text{Income from second stock} = \text{Rs.}\left[(188000 - x)\frac{5}{99}\right] \quad \text{... (ii)}$$

It is given that (i) = (ii)

$$\therefore \quad \frac{6x}{88} = \frac{(188000 - x) \times 5}{99}$$

$$\text{or} \quad 27\,x = 188000 \times 20 - 20\,x$$

$$\therefore \quad 47x = 188000 \times 20$$

$$\Rightarrow x = 80000$$

∴ Investment in 6 % stock at 88 = Rs. 80000

and, investment in 5 % stock at 99 = Rs. 108000

Example 37.17 Which is the better investment 14% stock at 95 or 15% stock at 105

Solution : Let the investment in each case by Rs. (95 × 105)

$$\text{Income in I case} = \text{Rs.}\ \frac{14}{95}(95 \times 105) = \text{Rs.}1470$$

$$\text{Income in II case} = \text{Rs.}\ \frac{15}{105}(95 \times 105)$$

$$= \text{Rs. } 1425$$

∴ Income in I case > Income in II case

∴ Ist case is a better investment

CHECK YOUR PROGRESS 37.2

1. Find the income on
   (a) 8 % stock of Rs. 50000 purchased at Rs. 110
   (b) 16% stock of Rs. 14000 purchased at Rs. 130
2. Find the income received by investing
   (a) Rs. 10000 in 11 % stock at 110
   (b) Rs. 36900 in 15% stock at 123
3. Renuka invested Rs. 58000 at Rs. 115 in 10% stock. Find her income if the brokerage is Re. 1
4. How much money should be invested to get an income of Rs. 3750 from $7\frac{1}{2}$% stock at 90 (Brokerage 2 %)

5. A man sells Rs. 6250, 8 % stock at 104 and invests the proceeds in 12% stock at 130. How much stock does the man hold now ?

6. A man bought Rs. 12500 of 8 % stock at Rs. 95 and sold it when it rose to Rs. 107. Find his total gain and gain percent [Brokerage : Re. 1]

7. Which of the following is a better investment ?

   (i) 9 % stock at 91 or 12 % stock at 121

   (ii) 11 % stock at 110 or 5 % stock at 60

8. A person invested Rs. 18000 in 8 % at 90 and sold the stock at 95 and invested the proceeds in 12 % stock, increasing his income by Rs. 680. At what price did he buy the latter stock ?

9. Vandana invested Rs. 12000 partly in 3 % stock at 75 and the remaining in 4 % stock at 96. If the total income from both the investments is Rs. 492, find the investment in each stock.

37.9 DEBENTURES

The capital is not only raised through shares, it is sometimes raised through loans, taken in the form of debentures.

A debenture is a written acknowledgment of a debt taken by a company. It contains a contract for the repayment of principal sum by some specific date and payment of interest at a specified rate irrespective of the fact, whether the company has a profit or loss. Debenture holders are, therefore, creditors of the company. Of course, they do not have any right on the profits declared by the company. Like shares, debentures can also be sold in or purchased from the market and all the terms used for shares also apply in this case ; with the same meanings.

Let us take some examples.

Example 37.18 Find the income percent of a buyer on 10 % debentures of face value Rs.100, available in the market at Rs. 125.

Solution : Income on Rs. 125 is Rs. 10

$$\therefore \qquad \text{Income on Rs. } 100 = \frac{10}{1255} \times 100 = \text{Rs. } 8$$

$$\therefore \qquad \text{Income, in percents, on debentures} = 8\%$$

Example 37.19 Shama has 1000 shares of par value Rs. 10 each of a company and 200 debentures of par value Rs. 100 each. The company pays an annual dividend of 10% and an interest of 15% on debentures. Find the total income of Shama and rate of return on her investment.

Solution : Dividend on 1000 shares $= \text{Rs.} \left(\frac{1000 \times 10 \times 10}{100} \right) = \text{Rs. } 1000$

Annual interest on 200 debentures $= \text{Rs.} \left(\frac{200 \times 100 \times 15}{100} \right) = \text{Rs. } 3000$

$\therefore$ Total income of Shama = Rs. 4000

Total investment of Shama = Rs. (1000 × 10 + 200 × 100) = Rs. 30000

$$\therefore \quad \text{Rate of return} = \left(\frac{4000 \times 100}{30000}\right)\% = 13.33\ \%$$

CHECK YOUR PROGRESS 37.3

1. Find the percent income on 10 % debentures of face value Rs. 120 available in the market for Rs. 150
2. Find the income percent on 10% debentures of face value Rs. 90 and available in the market for Rs. 120
3. Rama has 500 shares of par value Rs. 10 each and 100 debentures of par value Rs. 100 each of a company. The company pays an annual dividend of 12 % on the shares and 15% interest on debentures. Find the total income of Rama and her rate of interest on her investment.

LET US SUM UP

- Some like minded people join together, to start a big industry and form a Joint Stock company, a Company registered under Companies Act
- Shares are some convenient parts of the capital usually of Rs. 10 or Rs. 100 denomination.
- The Company invites the public to invest by purchasing shares of the company through public notices. Before that the company issues a prospectus of the company which details the objectives, its plus - points and weaknesses and then invites public to invest in the company.
- An individual who purchases/holds the share/shares of the company is called its shareholder.
- The company issues a share certificate to the share holder indicating the number of shares purchased and the par value of each share.
- The original value of the share fixed by the company is called its par value/nominal value/ face value.
- The part of profit distributed by the company to its shareholders is called the dividend.
- Preferred shares are those on which a fixed percent of dividend is to be paid, if the profit is left after paying for interest, working expenses, taxes, etc.
- Ordinary shares are the shares on which dividend is paid only after it is paid to preferred share holders. For that reason there is no fixed percentage of dividend to ordinary share holders
- The shares, as other commodities, can also be purchased from/sold in the market. This is done through an agent called broker and his charge is called brokerage. The brokerage is

generally expressed as percent of the par value of a share or fixed charge per share. The value at which a share can be sold/purchased form the market is called its market value.

- If the market value is greater than face value, the share is said to be at premium (or above par). If market value is less than face value the share is said to be below par (or at discount). If market value of a share equals its face value, it is said to be at par.
- Stock = Par value of a share × Number of shares.
- Stocks are referred to by their rates of dividend
- Income on stocks in different cases can be calculated by the use of unitary method as concepts used.
- A debenture is a written acknowledgment of a debt taken by a company. This is issued by a pre-decided date and the interest on it is paid irrespective of the fact that the company is in loss or profit.
- Like shares, debentures can also be sold in or purchased from the market.

SUPPORTIVE WEB SITES

1 http:// www.wikipedia.org.
1 http:// mathworld.wolfram.com

TERMINAL EXERCISE

1. Find the dividend received annually on 1250 shares of par value Rs. 10 each at 12 % payable annually.
2. Find the dividend received annually on 250 shares of par value Rs.100 each at 6 % payable semi-annually.
3. A company with 20000 shares of par value of Rs.50 each, shows a profit of Rs. 300000. If the dividend declared by the company is 5 %, then find.
   (i) the total dividend paid by the company
   (ii) the dividend received by a person holding 200 shares
4. A man buys 500 shares of a company, of par value Rs. 100 each, at Rs.150 each and sells each share at a premium of Rs. 75 each. Find his gain.
5. A man purchased 200 shares of a company, of par value of Rs.10 each paying an annual dividend of 12% at such a price that the gets 10% on his investment. Find the market price of a share.
6. Rama sells 2000 shares of a company A, of par value Rs.100 each paying a dividend of 12%, at Rs.126. She invested the proceed in another company B's shares of par value 25, at Rs. 30 each and giving 20% dividend. Find
   (i) the number of shares of company B purchased by Rama
   (ii) change in her dividend income

7. (i) Find the income on 6 % stock of Rs.60000 purchased at Rs.120

   (ii) Find the income on 16% of Rs. 14000 purchased at Rs. 130

8. Find the income received by investing

   (i) Rs. 9595 in 7 % stock at 95 (ii) Rs. 36900 in 15 % stock at 123

9. How much money should I invest to get an income of Rs.4500 from 10% stock at 94 [Brokerage ; Re. 1]

10. A man bought Rs. 12450 of 7 % stock at 95 and sold it when its price rise to Rs. 107. How much stock does the man hold now (Brokerage ; Re.1)

11. Which is the better investment

    (i) 10% stock at 110 or 5 % stock at 60 ?

    (ii) 10% stock at 90 or 11% stock at 103 ?

12. A person invested Rs. 9000 in 9 % stock at 90 and sold the stock at 95. He invested the proceeds in 12% stock thereby increasing his income by Rs. 240. At what price did be buy the second stock ?

13. Yogesh invested Rs. 94640 partly in 8% stock at 104 and remaining in 5 % stock at 91. If his total income from both stocks is Rs.6240, find the respective investment in two stocks.

14. A invested Rs. 47120 partly in 5 % stock at 75 and remaining in 6 % stock at 96. If the income from two stocks is equal, find the respective investment in each stock.

15. Find the income percent of a buyer on 12% debentures of face value Rs. 100, brought from the market for Rs. 144

16. Ram Lal had 1200 shares of par value Rs. 10 each and 400 debentures of par value Rs.100 each of a company. The company pays an annual dividend of 10% on shares and 12% interest on debentures. Find the total annual income of Ram Lal and return on his income.

17. Find the income percent on 8 % debentures of face value Rs. 80, available in the market at Rs. 120.

18. A man invests half of his money in $13\frac{1}{2}\%$ stock at 120 and the other half in $10\frac{1}{2}\%$ stock at 90. Had he invested his money to buy equal amount of each stock, he would have got Rs. 4.50 less of income. Find his total investment.

19. A person holds 60 debentures of a company of par value of Rs.500 on which 15% interest is paid annually. When the price rises to Rs.600, the person sells them and investes the half of sale proceeds in 8 % stock at 90. He invests the other half in 10% stock at 80 of par value Rs.100 each.

    Find the change in income.

ANSWERS

CHECK YOUR PROGRESS 37.1

1. Rs. 540 2. Rs. 5000 3. 16 %

4. (i) 5.7 % (ii) Rs. 814

5. Rs.8000 6. Rs. 200 7. Gain : Rs. 420

8. (i)6000 shares, (ii) Rs. 2500 more

CHECK YOUR PROGRESS 37.2

1. (a) Rs. 4000 (b) Rs.2240

2. (a) Rs.1000 (b) Rs. 4500

3. Rs. 5000 4. Rs. 45900 5. Rs.5000 6. Rs. 1250 ; 10.42 %

7. (i) Second (ii) First

8. At par 9. Rs. 4800, Rs. 7200

CHECK YOUR PROGRESS 37.3

1. 8% 2. 7.5 % 3. Rs. 2100 ; 14 %

TERMINAL EXERCISE

1. Rs. 1500 2. Rs. 3000 3. (i) Rs. 50000 (ii) Rs. 500

4. Rs. 12500 5. Rs. 12 per share

6. (i) 8400 (ii) Rs. 18000

7. (i) Rs. 3600 (ii) Rs.2240

8. (i) Rs. 707 (ii) Rs.4500 9. Rs. 42750

10. Rs. 1245 more

11. (i) First (ii) First 12. At par

13. Rs. 47320 each 14. Rs. 22800 ; Rs. 24320 15. $8\frac{1}{3}\%$

16. Rs. 6000 ; $11\frac{7}{13}\%$ 17. $5\frac{1}{3}\%$

18. Rs. 15120 19. Rs. 650 loss

38

INDEX NUMBERS

Of the important statistical devices and techniques, Index Numbers have today become one of the most widely used for judging the pulse of economy, although in the beginning they were originally constructed to gauge the effect of changes in prices. Today we use index numbers for cost of living, industrial production, agricultural production, imports and exports, etc.

Index numbers are the indicators which measure percentage changes in a variable (or a group of variables) over a specified time. By saying that the index of export for the year 2001 is 125, taking base year as 2000, it means that there is an increase of 25% in the country's export as compared to the corresponding figure for the year 2000.

OBJECTIVES

After studying this lesson, you will be able to :

- define index numbers and explain their uses;
- identify and use the following methods for construction of index numbers :
  (i) aggregate method (ii) simple average of relative method; and
- explain the advantages of different methods of construction.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of commercial mathematics
- Measures of Central Tendency

38.1 INDEX NUMBERS-DEFINITION

Some prominent definitions, given by statisticians, are given below:

According to the Spiegel :

"An index number is a statistical measure, designed to measure changes in a variable, or a group of related variables with respect to time, geographical location or other characteristics such as income, profession, etc."

According to Patternson :

" In its simplest form, an index number is the ratio of two index numbers expressed as a percent. An index is a statistical measure, a measure designed to show changes in one variable or a group of related variables over time, with respect to geographical location or other characteristics".

According to Tuttle :

"Index number is a single ratio (or a percentage) which measures the combined change of several variables between two different times, places or situations".

We can thus say that index numbers are economic barometers to judge the inflation (increase in prices) or deflationary (decrease in prices) tendencies of the economy. They help the government in adjusting its policies in case of inflationary situations.

38.2 CHARACTERISTICS OF INDEX NUMBERS

Following are some of the important characteristics of index numbers :

- Index numbers are expressed in terms of percentages to show the extent of relative change
- Index numbers measure relative changes. They measure the relative change in the value of a variable or a group of related variables over a period of time or between places.
- Index numbers measures changes which are not directly measurable.

  The cost of living, the price level or the business activity in a country are not directly measurable but it is possible to study relative changes in these activities by measuring the changes in the values of variables/factors which effect these activities.

38.3 PROBLEMS IN THE CONSTRUCTION OF INDEX NUMBERS

The decision regarding the following problems/aspect have to be taken before starting the actual construction of any type of index numbers.

(i) Purpose of Index numbers under construction
(ii) Selection of items
(iii) Choice of an appropriate average
(iv) Assignment of weights (importance)
(v) Choice of base period

Let us discuss these one-by-one

38.3.1 Purpose of Index Numbers

An index number, which is designed keeping, specific objective in mind, is a very powerful tool. For example, an index whose purpose is to measure consumer price index, should not include wholesale rates of items and the index number meant for slum-colonies should not consider luxury items like A.C., Cars refrigerators, etc.

38.3.2 Selection of Items

After the objective of construction of index numbers is defined, only those items which are related to and are relevant with the purpose should be included.

38.3.3 Choice of Average

As index numbers are themselves specialised averages, it has to be decided first as to which average should be used for their construction. The arithmetic mean, being easy to use and calculate, is preferred over other averages (median, mode or geometric mean). In this lesson, we will be using only arithmetic mean for construction of index numbers.

38.3.4 Assignment of weights

Proper importance has to be given to the items used for construction of index numbers. It is universally agreed that wheat is the most important cereal as against other cereals, and hence should be given due importance.

38.3.5 Choice of Base year

The index number for a particular future year is compared against a year in the near past, which is called base year. It may be kept in mind that the base year should be a normal year and economically stable year.

38.4 USES OF INDEX NUMBERS

(i) Index numbers are economic barometers. They measure the level of business and economic activities and are therefore helpful in gauging the economic status of the country.

(ii) Index numbers measure the relative change in a variable or a group of related variable(s) under study.

(iii) Consumer price indices are useful in measuring the purchasing power of money, thereby used in compensating the employes in the form of increase of allowances.

38.5 TYPES OF INDEX NUMBERS

Index numbers are names after the activity they measure. Their types are as under :

Price Index : Measure changes in price over a specified period of time. It is basically the ratio of the price of a certain number of commodities at the present year as against base year.

Quantity Index : As the name suggest, these indices pertain to measuring changes in volumes of commodities like goods produced or goods consumed, etc.

Value Index : These pertain to compare changes in the monetary value of imports, exports, production or consumption of commodities.

38.6 CONSTRUCTION OF INDEX NUMBERS

Suppose one is interested in comparing the sum total of expenditure on a fixed number of commodities in the year 2003 as against the year 1998. Let us consider the following example.

Commodity	Price (per unit) (in Rupees)	
	1998	2003
Wheat	200	400
Petrol	25	36
Pulses	$12\frac{1}{2}$	24
Sugar	10	18
Cooking Oil	80	80
Cloth	40	50

Since all the commodities are in different units and their prices are not enlarged proportionally, we just cannot get an average for comparison. For that reason, we express the rates of all commodities in 1998 as 100 each and proportionally increase for the corresponding commodities for 2003.

Commodity	1990		2003	
	price	Index	Price	Index
Wheat	200	100	400	$\frac{400\times100}{200}=200$
Petrol	25	100	36	$\frac{100}{25}\times36=144$
Pulses	$12\frac{1}{2}$	100	24	$\frac{100}{12.5}\times24=192$
Sugar	10	100	18	$\frac{100}{10}\times18=180$
Cooking Oil	80	100	80	$\frac{100}{80}\times80=100$
Cloth	40	100	50	$\frac{100}{40}\times50=125$
Average		100	Average	$941\div6=156.83$

We find that the average number (Index) for 2003 is 156.83 as against 100 for the year 1998. We can say that the prices have gone up by 56.83% in the year 2003 as against 1998. This method is used for finding price index numbers.

38.7 METHODS OF CONSTRUCTING INDEX NUMBERS

Construction of index numbers can be divided into two types :

(a) Unweighted indices

(b) Weighted indices

In this lesson, we will discuss only the unweighted indices:

The following are the methods of constructing unweighted index numbers :

(i) Simple Aggregative method

(ii) Simple average of price relative method

38.7.1 Simple Aggregative Method

This is a simple method for constructing index numbers. In this, the total of current year prices for various commodities is divided by the corresponding base year price total and multiplying the result by 100.

$\therefore$ Simple Aggregative Price Index $P_{01} = \dfrac{\Sigma p_1}{\Sigma p_0} \times 100$

Where P_{01} = Current price Index number

Σp_1 = the total of commodity prices in the current year

Σp_0 = the total of same commodity prices in the base year.

Let us take an example to illustrate :

Example 38. 1 Construct the price index number for 2003, taking the year 2000 as base year

Commodity	**Price in the year**	**Price in the year**
	2000	**2003**
A	60	80
B	50	60
C	70	100
D	120	160
E	100	150

Solution : Calculation of simple Aggregative index number for 2003 (against the year 2000)

Commodity	**Price in 2000**	**Price in 2003**
	(in Rs) p_0	(in Rs.) p_1
A	60	80
B	50	60
C	70	100
D	120	160
E	100	150
Total	$\sum p_0 = 400$	$\sum p_1 = 550$

Here $\sum p_0 = 400, \sum p_1 = 550$

$$\therefore \quad P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100 = \frac{550}{400} \times 100$$

$$= \frac{275}{2} = 137.5$$

i.e. the price index for the year 2003, taking 2000 as base year, is 137.5, showing that there is an increase of 37.5% in the prices in 2003 as against 2000.

Example 38.2 Compute the index number for the years 2001, 2002, 2003 and 2004, taking 2000 as base year, from the following data :

Year	2000	2001	2002	2003	2004
Price	120	144	168	204	216

Solution : Price relatives for different years are

2000	$\frac{120}{120} \times 100 = 100$
2001	$\frac{144}{120} \times 100 = 120$
2002	$\frac{168}{120} \times 100 = 140$
2003	$\frac{204}{120} \times 100 = 170$
2004	$\frac{216}{120} \times 100 = 180$

∴ Price index for different years are :

Year	2000	2001	2002	2003	2004
Price-Index	100	120	140	170	180

Example 38.3 Prepare simple aggregative price index number from the following data :

Commodity	Rate Unit	Price (1995)	Price (2004)
Wheat	per 10 kg	100	140
Rice	per 10 kg	200	250
Pulses	per 10 kg	250	350
Sugar	per kg	14	20
Oil	per litre	40	50

(b) Weighted indices

In this lesson, we will discuss only the unweighted indices:

The following are the methods of constructing unweighted index numbers :

(i) Simple Aggregative method

(ii) Simple average of price relative method

38.7.1 Simple Aggregative Method

This is a simple method for constructing index numbers. In this, the total of current year prices for various commodities is divided by the corresponding base year price total and multiplying the result by 100.

$$\therefore \text{ Simple Aggregative Price Index } P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

Where P_{01} = Current price Index number

Σp_1 = the total of commodity prices in the current year

Σp_0 = the total of same commodity prices in the base year.

Let us take an example to illustrate :

Example 38.1 Construct the price index number for 2003, taking the year 2000 as base year

Commodity	Price in the year	Price in the year
	2000	2003
A	60	80
B	50	60
C	70	100
D	120	160
E	100	150

Solution : Calculation of simple Aggregative index number for 2003 (against the year 2000)

Commodity	Price in 2000 (in Rs) p_0	Price in 2003 (in Rs.) p_1
A	60	80
B	50	60
C	70	100
D	120	160
E	100	150
Total	$\sum p_0 = 400$	$\sum p_1 = 550$

Here $\sum p_0 = 400, \sum p_1 = 550$

$$\therefore \quad P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100 = \frac{550}{400} \times 100$$

$$= \frac{275}{2} = 137.5$$

i.e. the price index for the year 2003, taking 2000 as base year, is 137.5, showing that there is an increase of 37.5% in the prices in 2003 as against 2000.

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Year	2000	2001	2002	2003	2004
Price	120	144	168	204	216

Solution : Price relatives for different years are

2000	$\frac{120}{120} \times 100 = 100$
2001	$\frac{144}{120} \times 100 = 120$
2002	$\frac{168}{120} \times 100 = 140$
2003	$\frac{204}{120} \times 100 = 170$
2004	$\frac{216}{120} \times 100 = 180$

$\therefore$ Price index for different years are :

Year	2000	2001	2002	2003	2004
Price-Index	100	120	140	170	180

Example 38.3 Prepare simple aggregative price index number from the following data :

Commodity	Rate Unit	Price (1995)	Price (2004)
Wheat	per 10 kg	100	140
Rice	per 10 kg	200	250
Pulses	per 10 kg	250	350
Sugar	per kg	14	20
Oil	per litre	40	50

Solution : Calculation of simple aggregative index number.

Commodity	Rate Unit	Price (1995)	Price (2004)
Wheat	per 10 kg	100	140
Rice	per 10 kg	200	250
Pulses	per 10 kg	250	350
Sugar	per kg	14	20
Oil	per litre	40	50
		604	**810**

Simple Aggregative index number

$$= \frac{810}{604} \times 100 = 134.1$$

CHECK YOUR PROGRESS 38.1

1. Write the characteristics and uses of index numbers.
2. Enumerate the problems /aspects in the construction of index numbers.
3. Find the simple aggregative index number for each of the following :

(i) For the year 2000 with 1980 as base year

Commodity	Price in 1980	Price in 2000
A	200	250
B	110	150
C	20	30
D	210	250
E	25	25

(ii) For the years 1999, 2000, 2001, 2002, 2003 taking 1998 as base year

Year	1998	1999	2000	2001	2002	2003
Price	20	25	28	30	35	40

(iii) For the years 2001 and 2002 taking 1999 as base year.

Commodity		A	B	C	D	E	F
price in	1999	10	25	40	30	25	100
	2001	12	30	50	30	25	110
	2002	15	30	60	40	30	120

38.7.2 Simple Average of Price Relatives Method

In this method, the price relatives for all commodities is calculated and then their average is taken to calculate the index number.

Thus, $P_{01} = \dfrac{\Sigma \dfrac{p_1}{p_0} \times 100}{N}$, if A.M. is used as average where P_{01} is the price index, N is the number of items, p_0 is the price in the base year and p_1 of corresponding commodity in present year (for which index is to be calculated)

Let us take an example.

Example 38.4 Construct by simple average of price relative method the price index of 2004, taking 1999 as base year from the following data :

Commodity	**A**	**B**	**C**	**D**	**E**	**F**
Price (in 1999)	60	50	60	50	25	20
Price (in 2004)	80	60	72	75	$37\frac{1}{2}$	30

Solution :

Commodity	Price (in 1999) (in Rs.)[p_0]	Price (in 2004) (in Rs.) [p_1]	Price Relatives $\left(\frac{p_1}{p_0} \times 100\right)$
A	60	80	133.33
B	50	60	120.00
C	60	72	120.00
D	50	75	150.00
E	25	$37\frac{1}{2}$	150.00
F	20	30	150.00
			823.33

$$\therefore \quad P_{01} = \frac{\Sigma \frac{p_1}{p_0} \times 100}{N} = \frac{823.33}{6} = 137.22$$

$\therefore$ Price index for 2004, taking 1999 for base year = 137.22

Example 38.5 Using simple average of Price Relative Method find the price index for 2001, taking 1996 as base year from the following data :

Solution : Calculation of simple aggregative index number.

Commodity	Rate Unit	Price (1995)	Price (2004)
Wheat	per 10 kg	100	140
Rice	per 10 kg	200	250
Pulses	per 10 kg	250	350
Sugar	per kg	14	20
Oil	per litre	40	50
		604	**810**

Simple Aggregative index number

$$= \frac{810}{604} \times 100 = 134.1$$

CHECK YOUR PROGRESS 38.1

1. Write the characteristics and uses of index numbers.
2. Enumerate the problems /aspects in the construction of index numbers.
3. Find the simple aggregative index number for each of the following :

(i) For the year 2000 with 1980 as base year

Commodity	Price in 1980	Price in 2000
A	200	250
B	110	150
C	20	30
D	210	250
E	25	25

(ii) For the years 1999, 2000, 2001, 2002, 2003 taking 1998 as base year

Year	1998	1999	2000	2001	2002	2003
Price	20	25	28	30	35	40

(iii) For the years 2001 and 2002 taking 1999 as base year.

Commodity		A	B	C	D	E	F
price in	1999	10	25	40	30	25	100
	2001	12	30	50	30	25	110
	2002	15	30	60	40	30	120

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In this method, the price relatives for all commodities is calculated and then their average is taken to calculate the index number.

Thus, $P_{01} = \dfrac{\Sigma \dfrac{p_1}{p_0} \times 100}{N}$, if A.M. is used as average where P_{01} is the price index, N is the number of items, p_0 is the price in the base year and p_1 of corresponding commodity in present year (for which index is to be calculated)

Let us take an example.

Example 38.4 Construct by simple average of price relative method the price index of 2004, taking 1999 as base year from the following data :

Commodity	**A**	**B**	**C**	**D**	**E**	**F**
Price (in 1999)	60	50	60	50	25	20
Price (in 2004)	80	60	72	75	$37\frac{1}{2}$	30

Solution :

Commodity	Price (in 1999) (in Rs.) [p_0]	Price (in 2004) (in Rs.) [p_1]	Price Relatives $\left(\frac{p_1}{P_0} \times 100\right)$
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F	20	30	150.00
			823.33

$$\therefore \quad P_{01} = \frac{\Sigma \dfrac{p_1}{p_0} \times 100}{N} = \frac{823.33}{6} = 137.22$$

$\therefore$ Price index for 2004, taking 1999 for base year = 137.22

Example 38.5 Using simple average of Price Relative Method find the price index for 2001, taking 1996 as base year from the following data :

Commodity	Wheat	Rice	Sugar	Ghee	Tea
Price (in 1996) per unit	12	20	12	40	80
Price (in 2001) per unit	16	25	16	60	96

Solution :

Commodity	Price (in 1996) (in Rs.) $[p_0]$	Price (in 2001) (in Rs.) $[p_1]$	Price Relatives $\left(\frac{p_1}{p_0}\times 100\right)$
Wheat	12	16	$\frac{16}{12}\times 100 = 133.33$
Rice	20	25	$\frac{25}{20}\times 100 = 125.00$
Sugar	12	16	$\frac{16}{12}\times 100 = 133.33$
Ghee	40	60	$\frac{60}{40}\times 100 = 150.00$
Tea	80	96	$\frac{96}{80}\times 100 = 120.00$
			661.66

$$P_{01} = \frac{\Sigma \frac{p_1}{p_0}\times 100}{N}$$

$$= \frac{661.66}{5} = 132.33$$

$\therefore$ Price Index for 2001, taking 1996 as base year, = 132.33

CHECK YOUR PROGRESS 38.2

Using Simple Average of Relatives Method, find price index for each of the following :

(i) For 2004, taking 2000 as base year

Commodity	A	B	C	D	E
Price in 2000	15	16	60	40	20
Price in 2004	20	20	80	50	25

(ii) For 2001, taking 1999 as base year

Commodity	**Wheat**	**Rice**	**Sugar**	**Ghee**	**Tea**
Price (per unit) in 1999	10	20	60	40	16
Price (per unit) in 2001	12	22	80	50	20

LET US SUM UP

- An index number is a statistical measure, designed to measure changes in a variable(s) with time/geographical location/other criteria
- Index Numbers are of three types : (i) Price-Index Numbers
  (ii) Quantity Index Numbers
  (iii) Value-index Numbers
- Method of construction of Index numbers

(i) Simple Aggregative method $P_{01} = \Sigma \frac{p_1}{p_0} \times 100$

where P_{01} is the price index

p_0 is the price of a commodity in base year

p_1 is the price of the commodity in present year

(ii) Simple Average of Price Relatives Method

$$P_{01} = \frac{\Sigma \frac{p_1}{p_0} \times 100}{N}$$

Where N is the number of commodities and all others as in (i) above.

SUPPORTIVE WEB SITES

http :// www.wikipedia.org

http :// mathworld.wolfram.com

TERMINAL EXERCISE

1. Use Simple Aggregative Method, find the price index for each of the following :

Commodity	**Wheat**	**Rice**	**Sugar**	**Ghee**	**Tea**
Price (in 1996) per unit	12	20	12	40	80
Price (in 2001) per unit	16	25	16	60	96

Solution :

Commodity	Price (in 1996) (in Rs.) $[p_0]$	Price (in 2001) (in Rs.) $[p_1]$	Price Relatives $\left(\frac{p_1}{P_0} \times 100\right)$
Wheat	12	16	$\frac{16}{12} \times 100 = 133.33$
Rice	20	25	$\frac{25}{20} \times 100 = 125.00$
Sugar	12	16	$\frac{16}{12} \times 100 = 133.33$
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∴ Price Index for 2001, taking 1996 as base year, = 132.33

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Commodity	A	B	C	D	E
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Where N is the number of commodities and all others as in (i) above.

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1. Use Simple Aggregative Method, find the price index for each of the following :

(i) For the year 2000, taking 1990 as base year

Commodity	A	B	C	D	E
Price (in Rs.) in 1990	10	14	18	20	100
Price (in Rs.) in 2000	12	20	20	25	110

(ii) For the year 2004, taking 1998 as base year

Commodity	A	B	C	D	E	F
Price in 1998	20	28	110	80	60	20
Price in 2004	25	40	120	100	80	25

(iii) For the year 1996, 1997, 1998, 1999, Taking 1990 as base year

Commodity		A	B	C	D
Price in	1990	5	8	10	22
	1996	10	12	20	18
	1997	12	15	20	16
	1998	10	15	25	22
	1999	15	20	28	22

2. Using Simple Average of Price Relative Method, find the price index for each of the following:

(i) For 2000, taking 1998 as base year

Commodity	A	B	C	D	E
Price (in Rs.) 1998	12	20	24	28	20
Price (in Rs.) 2000	16	25	30	35	26

(ii) For the year 2004, taking 1999 as base year

Commodity	A	B	C	D	E	F
Price (in Rs.) in 1999	12	28	32	36	40	50
Price (in Rs.) in 2004	16	35	40	45	50	60

(iii) For the years 2003 and 2004 Taking 1998 as base year

Commodity	A	B	C	D
Price (in Rs.) in 1998	4	28	30	40
Price (in Rs.) in 2003	5	35	36	50
Price (in Rs.) in 2004	6	42	42	65

ANSWERS

CHECK YOUR PROGRESS 38.1

3. (i) 124.78

(ii) 1999:125; 2000:140; 2001:150; 2002:175; 2003:200

(iii) 11.74 ; 128.26

CHECK YOUR PROGRESS 38.2

(i) 128.33 (ii) 122.67

TERMINAL EXERCISE

1. (i) 115.43 (ii) 122.64

(iii) 1996: 133.33 ; 1997: 140.00 ; 1998 : 160.0 ; 1999:188.88

2 (i) 127.67 (ii) 125.56 (iii) 2003:123.75 ; 2004 :150.625

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39

INSURANCE

It is a commonly acknowledged phenomenon that there are countless risks in every sphere of life. For property, there are fire risks; for shipment of goods, there are perils of sea; for human life, there are risks of death or disability; and so on. The chances of occurrences of the events causing losses are quite uncertain because these may or may not take place. In other words, our life and property are not safe and there is always a risk of losing it. A simple way to cover this risk of loss money-wise is to get life and property insured. In this business, people facing common risks come together and make their small contributions to the common fund. While it may not be possible to tell in advance, which person will suffer the losses, it is possible to work out how many persons on an average out of the group may suffer the losses.

When risk occurs, the loss is made good out of the common fund. In this way, each and every one shares the risk. In fact, insurance companies bear risk in return for a payment of premium, which is calculated on the likelihood of loss. In this lesson, you will learn Insurance, its various kinds, premium calculation, calculation of paid up/surrender value etc in details.

OBJECTIVES

After studying this lesson, you will be able to :

- define insurance and its significance;
- define various kinds of insurance-General Insurance, Life Insurance;
- explain the scope of insurance;
- define various schemes (Group Insurance scheme, Retirement scheme etc.) and policies of insurances;
- calculate premiums of various types of insurance;
- calculate the age of life to be assured;
- calculate actual premium; and
- calculate paid up/surrender value keeping policy inforce.

EXPECTED BACKGROUND KNOWLEDGE

- Percentage and its application

39.1 WHAT IS AN INSURANCE?

Insurance is a tool by which fatalities of a small number are compensated out of funds collected from the insured. Insurance companies pay back for financial losses arising out of occurrence of insured events, e.g. in personal accident policy the insured event is death due to accident, in fire policy the insured events are fire and other natural calamities. Hence, insurance is a safeguard against uncertainties. It provides financial recompense for losses suffered due to incident of unanticipated events, insured within the policy of insurance.

Insurance, essentially, is an arrangement where the losses experimented by a few are extended over several who are exposed to similar risks. Insurance is a protection against financial loss arising on the happening of an unexpected event.

An individual who wants to cover risk pays a small amount of money to an organization called on Insurance Company and gets insured. An insurance company insures different people by collecting a small amount of money from each one of them and collectively this money is enough to compensate or cover the loss that some members may suffer.

The fixed amount of money paid by the insured to the insurance company regularly is called premium. Insurance company collects premium to provide security for the purpose.

Insurance is an agreement or a contract between the insured and the Insurance Company (Insurer).

39.2 NATURE OF INSURANCE

On the basis of the definition of insurance discussed above, one can observe its following characteristics:

39.2.1 Risk Sharing and Risk Transfer

Insurance is a mechanism adopted to share the financial losses that might occur to an individual or his family on the happening of a specified event. The event may be death of earning member of the family in the case of life insurance, marine-perils in marine insurance, fire in fire insurance and other certain events in miscellaneous insurance, e.g., theft in burglary insurance, accident in motor insurance, etc. The loss arising from these events are shared by all the insured in the form of premium. Hence, risk is transferred from one individual to a group.

39.2.2 Co-operative Device

Insurance is a cooperative device under which a group of persons who agree to share the financial loss may be brought together voluntarily or through publicity or through solicitations of the agents. An insurer would be unable to compensate all the losses from his own capital. So, by insuring a large number of persons, he is able to pay the amount of loss.

39

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Notes

Example 39.1 In a village, there are 1000 houses. Each house is valued at Rs.30,000 on an average. If 10 houses get burnt every year, calculate the total loss per year. Calculate how much money each house owner should contribute per year to compensate total loss caused by fire.

Solution : Total loss per year $= Rs.30000 \times 10 = Rs.300000$

$$\text{Required contribution from each house owner} = Rs.\ \frac{300000}{1000} = Rs.\ 300$$

$\therefore$ All the 1000 house owners should agree to contribute a sum of Rs. 300 each at the beginning of the year and create a fund. This will be enough to pay a compensation of Rs. 30,000 to each of the 10 house owners whose house are burnt by fire. This way the risk of loss of 10 house owners is spread over a group of 1000 house owners.

Example 39.2 In a town, there are 10,000 persons who are all aged 50 years and are enjoying normal health. It is expected that 20 persons may die during the year. If the economic value of the loss suffered by the family of each dying person were taken to be Rs. 50,000, calculate the total loss. How much money each person should contribute to compensate the total loss?

Solution: Total loss workout to be $= Rs.\ 50000 \times 20 = Rs.\ 1000000$

$$\text{Required contribution from each person/year would be} = Rs.\ \frac{1000000}{10000} = Rs.\ 100$$

This amount is enough to pay Rs. 50,000 to the family of each of the 20 dying persons. Thus, the risk of 20 persons are shared by 10,000 persons.

39.3 BRIEF HISTORY OF INSURANCE

Marine insurance is the oldest form of insurance followed by life insurance and fire insurance. The history of insurance can be traced back to the early civilization. As civilization progressed, the incidence of losses started increasing giving rise to the concept of loss sharing. The Aryans through their village co-operatives practiced loss of profit insurance. The code of Manu indicates that there was a practice of marine insurance carried out by the traders in India with those of Sri Lanka, Egypt and Greece.

The earliest transaction of insurance as practiced today can be traced back to the 14th century A.D. in Italy when ship were only being covered. This practice of Marine Insurance, gradually spread to London during 16th century. The history of Marine Insurance is closely linked with the origin and rise of the Lloyd's shipowners. Marine traders, who used to gather at Lloyd's coffee house in London, agree to share losses to goods during transportation by ship.

Marine related losses included:

- Loss of ship by sinking due to bad weather
- Goods in transit by ship robbed by sea pirates
- Loss or damage to the goods in transit by ship due to bad weather in high sea.

The Lloyd's Act was framed to set up the Lloyd's by whom they were empowered to transact other classes of Insurance. Today, Lloyd's is regarded as the largest insurance underwriter in the world. The first insurance policy was issued in England in 1583.

39.4 TYPES OF INSURANCE

Insurance occupies an important place in the modern world because the risk, which can be insured, have increased in number and extent owing to the growing complexity of the present day economic system. It plays a vital role in the life of every citizen and has developed on an enormous scale leading to the evolution of many different types of insurance. In fact, now a day almost any risk can be made the subject matter of contract of insurance. The different types of insurance have come about by practice within insurance companies, and by the influence of legislation controlling the transacting of insurance business. Broadly, insurance may be classified into the following categories:

(1) Classification on the basis of nature of insurance
  (a) Life Insurance
  (b) Fire Insurance
  (c) Marine Insurance
  (d) Social Insurance
  (e) Miscellaneous Insurance

(2) Classification from business point of view:
  (a) Life Insurance
  (b) General Insurance

(3) Classification from risk point of view:
  (a) Personal Insurance
  (b) Property Insurance
  (c) Liability Insurance
  (d) Fidelity Guarantee Insurance

However, in the present lesson we will discuss insurance in business point of view, personnel insurance and property insurance.

39.5 LIFE INSURANCE IN INDIA

In India, insurance started with life insurance. It was in the early 19th century when the Britishers on their postings in India felt the need of life insurance cover.

It started with English Companies like. 'The European and the Albert'. The first Indian insurance company was the Bombay Mutual Insurance Society Ltd., formed in 1870.

In the wake of the Swadeshi Movement in India in the early 1900s; quite a good number of Indian companies were formed in various parts of the country to transact insurance business. To name a few: 'Hindustan Co-operative' and 'National Insurance' in Kolkata; 'United India' in Chennai; 'Bombay Life', 'New India' and 'Jupiter' in Mumbai and 'Lakshmi Insurance' in New Delhi.

39.6 WHY LIFE INSURANCE?

It Covers the Risk of Death

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39.3 BRIEF HISTORY OF INSURANCE

Marine insurance is the oldest form of insurance followed by life insurance and fire insurance. The history of insurance can be traced back to the early civilization. As civilization progressed, the incidence of losses started increasing giving rise to the concept of loss sharing. The Aryans through their village co-operatives practiced loss of profit insurance. The code of Manu indicates that there was a practice of marine insurance carried out by the traders in India with those of Sri Lanka, Egypt and Greece.

The earliest transaction of insurance as practiced today can be traced back to the 14th century A.D. in Italy when ship were only being covered. This practice of Marine Insurance, gradually spread to London during 16th century. The history of Marine Insurance is closely linked with the origin and rise of the Lloyd's shipowners. Marine traders, who used to gather at Lloyd's coffee house in London, agree to share losses to goods during transportation by ship.

Marine related losses included:

- Loss of ship by sinking due to bad weather
- Goods in transit by ship robbed by sea pirates
- Loss or damage to the goods in transit by ship due to bad weather in high sea.

The Lloyd's Act was framed to set up the Lloyd's by whom they were empowered to transact other classes of Insurance. Today, Lloyd's is regarded as the largest insurance underwriter in the world. The first insurance policy was issued in England in 1583.

Notes

39.4 TYPES OF INSURANCE

Insurance occupies an important place in the modern world because the risk, which can be insured, have increased in number and extent owing to the growing complexity of the present day economic system. It plays a vital role in the life of every citizen and has developed on an enormous scale leading to the evolution of many different types of insurance. In fact, now a day almost any risk can be made the subject matter of contract of insurance. The different types of insurance have come about by practice within insurance companies, and by the influence of legislation controlling the transacting of insurance business. Broadly, insurance may be classified into the following categories:

(1) Classification on the basis of nature of insurance
  (a) Life Insurance
  (b) Fire Insurance
  (c) Marine Insurance
  (d) Social Insurance
  (e) Miscellaneous Insurance

(2) Classification from business point of view:
  (a) Life Insurance
  (b) General Insurance

(3) Classification from risk point of view:
  (a) Personal Insurance
  (b) Property Insurance
  (c) Liability Insurance
  (d) Fidelity Guarantee Insurance

However, in the present lesson we will discuss insurance in business point of view, personnel insurance and property insurance.

39.5 LIFE INSURANCE IN INDIA

In India, insurance started with life insurance. It was in the early 19th century when the Britishers on their postings in India felt the need of life insurance cover.

It started with English Companies like. 'The European and the Albert'. The first Indian insurance company was the Bombay Mutual Insurance Society Ltd., formed in 1870.

In the wake of the Swadeshi Movement in India in the early 1900s; quite a good number of Indian companies were formed in various parts of the country to transact insurance business. To name a few: 'Hindustan Co-operative' and 'National Insurance' in Kolkata; 'United India' in Chennai; 'Bombay Life', 'New India' and 'Jupiter' in Mumbai and 'Lakshmi Insurance' in New Delhi.

39.6 WHY LIFE INSURANCE?

It Covers the Risk of Death

Notes

The risk of death is covered under insurance scheme but not under ordinary savings plans. In case of death, insurance pays full sum assured, which would be several times larger than the total of the premiums paid. Under ordinary savings plans, only accumulated amount is payable.

It Encourages Compulsory Saving

After taking insurance, if the premium is not paid, the policy lapses. Therefore, the insured is forced to go on paying premium. In other words it is compulsory. A savings deposit can be withdrawn very easily.

Easy Settlement and Protection against Creditors

Once nomination or assignment is made, a claim under life insurance can be settled in a simple way. Under M.W.P. Act, the policy moneys become a kind of trust, which cannot be taken away, even by the creditors.

It helps to Achieve the Purpose of the Life Assured

If a lump sum amount is received in the hands of anybody, it is quite likely that the amount might be spent unwisely or in a speculative way.

To overcome this risk, the life assured can provide that the claim amount be given in instalments.

Peace of Mind

The knowledge that insurance exists to meet the financial consequences of certain risks provides a form of peace of mind. This is important for private individuals when they insure their car, house, possessions and so on, but it is also vital importance in industry and commerce.

Loss Control

Insurance is primarily concerned with the financial consequences of losses, but it would be fair to say that insurers have more than a passing interest in loss control. It could be argued that insurers have no real interest in the complete control of loss, because this would inevitably lead to an end to their business.

Social Benefits

The fact that the owner of a business has the funds available to receiver from a loss provides the stimulus to business activity we noted earlier. It also means that jobs may not be lost and goods or services can still be sold. The social benefit of this is that people keep their jobs, their sources of income are maintained and they can continue to contribute to the national economy.

Investment of Funds

Insurance companies have at their disposal large amounts of money. This arises from the fact that there is a gap between the receipt of a premium and the payment of a claim. A premium could be paid in January and a claim may not occur until December, if it occurs at all. The insurer has this money and can invest it.

Invisible Earnings

We have already said that insurance allows people and organizations to spread risk among them. In the same way, we can also say that countries spread risk. A great deal of insurance is transacted in the UK in respect of property and liabilities incurred overseas. London is still very much the centre of world insurance and large volumes of premium flow into London every year; these are described as invisible earnings.

Insurance Facilitates Liquidity

If a policyholder is not in a position to pay the premium, he can surrender the policy for a cash sum.

Loan Facility and Tax Relief

The person can also take a loan for a temporary period to tide over the difficulty. Sometimes, a life insurance policy is acceptable as security for a commercial loan. By paying the insurance premium, the insured obtains significant reliefs in Income Tax and Wealth Tax.

39.7 NATIONALIZATION OF LIFE INSURANCE IN INDIA

In 1956, life insurance business was nationalized and LIC of India came into being on 01-09-1956. The government took over the business of 245 companies (including 75 provident fund societies) who were transacting life insurance business at that time. Thereafter, LIC got the exclusive privilege to transact life insurance business in India.

Relevant laws were amended in 1999 and LIC's monopoly right to transact life insurance business in India came to an end. At the close of financial year ending 31-03-2004, twelve new companies were registered with the Insurance Regulatory and Development Authority (IRDA) to transact life insurance business in India.

39.8 LIFE INSURANCE CORPORATION (LIC)

As the name suggests, Life Insurance is an insurance of the life of an individual. Thus, life insurance is a contract between the insured and the insurer i.e., the Life Insurance Corporation (LIC). The written record of this contract is called an Insurance policy.

The person who is specified by the insured to receive the insurance policy in case premature death is called a Nominee.

Let us study some types of insurance policies:

1. Whole Life Policy

In some policies, premium is to be paid throughout the life-time of the insured. The payment of premium stops on the death of the insured and the money i.e., amount of the policy is paid to

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2. Endowment Policy

In some policies premium is paid for a fixed time period and the amount of the policy is paid to the insured after this time period.

The date on which the amount of a policy becomes due is called the date of maturity and the time period for which the insurance is taken is called Endowment Term.

In case, the insured dies before the date of maturity, the payment of premium is stopped immediately and the nominee gets the amount of the policy.

Such a policy is called the Endowment Insurance Policy.

There are many policies under both these categories. Some of the most popular ones are Jeevan Dhara, Jeevan Mitra, Jeevan Sarita, Money Back Policy, Jeevan Kishore etc.

Some of the policies are said to be **'with profits'** and some **'without profits'.**

The policy-holders (i.e. insured) who have a policy 'with profits' share the profits of the LIC. The LIC pays a part of its profits called bonus (as a percentage of the amount of the policy) to such policy – holders.

The policy holders who have policies without profits are not paid this bonus. The premium in case of policies with profits, is generally higher than the premium of policies 'without profits'.

3. Group Savings – Linked Insurance Scheme (GIS)

This insurance scneme is offered to a group of salaried employees of State/Central Government Undertakings. The scheme is also available to reputed limited companies subject to certain conditions being satisfied.

Rate of premium is much lower in group insurance as compared to LIC.

The employees are grouped into categories based on their designations.Maximum risk cover available under the scheme is given in the follow:

Group Size	Category	Maximum Cover
50-99	A	Rs. 80,000
	B	Rs. 60,000
	C	Rs. 40,000
	D	Rs. 20,000
100 and above	A	Rs. 1,20,000
	B	Rs. 90,000
	C	Rs. 60,000
	D	Rs. 30,000

The monthly premium paid towards GIS is Rs. 10 for an insurance cover of Rs. 10,000 where in Rs. 10 covers insurance premium and savings both. The savings is 65% and insurance premium is 35% of the amount paid towards the GIS i.e. out of Rs. 10, Rs. 6.50 is the savings and Rs. 3.50 is the insurance premium of the individual. Compound interest at a fixed rate of 8% is paid on the savings in GIS.

In case of death of the individual during service, the nominee gets both the full Insurance money and the savings with interest.

In this lesson, we will now learn how to calculate the premium using the Tables (See Annexure-A) in the life insurance policies. The tables give the premium for an insurance of Rs. 1000 at a particular age.

39.9 CALCULATION OF AGE OF THE LIFE TO BE ASSURED

Risk of death is closely related to the age of the life to be assured. Hence, the age at entry into the contract of insurance becomes the most significant factor to determine premium. Months and days over the completed years of age are not taken as such, but the age to be taken is rounded off to the years in integer may be defined as :

(1) Age nearer to the birthday
(2) Age on next birthday
(3) Age on last birthday

If a person was born 22 years 8 months earlier. Then

(1) Age nearer to the birthday is 23
(2) Age on next birthday is 23
(3) Age as on last birthday is 22

If a person is 22 years 5 months 29 days then the age nearer birthday will be 22 years and if the age is 22 years 5 months 30 days the age nearer birthday will be 23 years.

We will explain this with the help of the following examples:

(1) If a person is born on 1/1/1980, then on 1/8/2000 he is 20 years 7 months and 1 day old therefore:

(a) his age nearer to his birthday	21 years.
(b) his age as per last birthday	20 years.
(c) his age as per next birthday	21 years.

(2) If a person is born on 1/1/1980, then on 11/4/2000 he is 20 years 3 months and 11 days old therefore:

(a) his age nearer to his birthday	20 years.
(b) his age as per last birthday	20 years.
(c) his age as per next birthday	21 years.

39.10 CALCULATION OF ACTUAL PREMIUM

The **"Tables of Premiums"** (See Annexure-'A') prescribed by various life insurance companies in India, show their premium amount per thousand per year. However, some companies have adopted half yearly, quarterly mode of payment as the basis, while others have adopted 'yearly' mode as the basis.

Notes

the nominee. This type of policy is called the whole life policy.

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A grace period of 30 is allowed sometimes to make payment of premium in case of yearly, half yearly or quarterly payment and upto 15 days grace period is allowed in case of monthly payment of premium.

After calculating the age, the premium will be calculated as follows:

(1) Tabular premium for the age concerned

(2) Loading proposal for reason of health and/or physical impairments. Extras on adverse health features or adverse Medical report e.g. Blood pressure, sugar, diabetic, smokers, etc.

(3) **Extra for occupation :** There are extra premium on hazardous or extra-hazardous occupations e.g. Aviation and defence, mining and other occupational risks.

(4) Extra for accident benefits (if asked and if allowed): To get additional benefit on account of accidental death, the extra premium is to be paid for **Double Accident Benefit (DAB) and Extended Permanent Disability Benefit (EPDB).**

(5) **Extra for premium waiver benefit:** If a person becomes disabled then he will not be able to pay the premium because he may not be able to earn because of his disability. Therefore, the company waives off the premium on payment of additional premium.

(6) **Mode of Payment:** Adjustment are made for different mode of payment as per details given below:

Mode	Rebates
1. Yearly	3% of Tabular premium
2. Half-Yearly	1.5% of Tabular premium
3. For Quarterly mode and Monthly mode under Salary Saving Scheme (SSS)	No Rebate : No loading
4. For Ordinary Monthly mode except Salary Saving Scheme for monthly payment	Loading of 5% on Tabular Premium

(7) **Rebate for large sum assured :** Adjustments are also made for higher sum assured.

For every new policy there are certain:

- **'fixed costs'** which are uniform for all policies irrespective of sum assured, for example, cost of policy preparation or postal expenses for mailing the policy document.
- **'variable costs'** depending on the sum assured; for example stamp duty on the policy document or medical examiner's fee.

When the sum assured is large, fixed costs get reduced per thousand sum assured resulting into savings to the insurer. Insurer shares these savings with the policy holders by offering rebate in tabular premium for large sum assured.

The reduction in premium for large sum assured ranges from Rs. 1 to Rs. 8 per thousand varying from company to company and the type of product.

39.11 ACCIDENT BENEFIT

LIC also offers policies with accident benefits. In such policies, if the insured gets a permanent disability due to accident or dies in an accident, the LIC pays double the sum assured. In such a policy, while calculating the premium, an extra amount of Re 1 per thousand per annum is added to the tabular premium.

Rounding off

If the paise portion of the premium is 0.50 or less, it is rounded off to the lower rupee and if it is more than 0.50, it is rounded off to the next higher rupee.

The different insurers follow different rates but the oldest Insurance Company in India, i.e. LIC follows the following discounts structure:

Rebates assumed for large Sum Assured:

Sum Assured Rebates

Sum Assured	Rebates
1. Upto Rs. 24,999	No Rebate
2. From Rs. 25,000 to Rs. 49,999	@ Re 1 per thousand sum assured
3. From Rs. 50,000 and above	@ Rs. 2 per thousand sum assured
Extra Premium to be charged for grant of Double Accident Benefit (DAB) and Extended Permanent Disability Benefit (EPDB)	@ Rs. 1 per thousand of sum assured

Where the premiums are payable on half yearly basis, there is saving in administrative expenses compared to quarterly mode. In half yearly or yearly mode the insurer issues less number of notices and fewer collection receipts and consequential accounting entries would also be less. This would result in saving in administrative cost. Moreover the insurer can earn more interest. While for monthly payment the extra premium is to be charged to cover up additional administrative expenses.

In short we can say:

Lesser number of installment of premium : Higher amount but more discount
More number of installment of premium : Lower Amount but less discount

Premium Amount=Sum Assured × Premium Rate / 1000

Let us now take some examples on calculation of premium, using the tables, and the rates of rebate etc. given above.

Example 39.3

(i) A person at the age of 25 years takes a insurance policy of sum assured Rs. 50,000 for 30 years term. Calculate premium for yearly payment assuming the following details:

Tabular Premium/1000	Rs. 40
Rebate for large Sum assured	Rs. 2 per 1000
Rebate for yearly payment	3%

Solution :

Premium Calculation	**Rs.**
Tabular Premium/1000	40.00
Less Rebate for large Sum Assured	−2.00
Less Rebate for yearly payment (3% of Rs. 40)	−1.20
Total	= 36.80

Yearly Premium = Rs. 36.80 × 50000/1000 = Rs. 1840

(ii) A person at the age of 25 years takes a insurance policy of sum assured Rs. 50,000 for 30 years term. Calculate premium for half-yearly payment assuming the following details:

Tabular Premium/1000	Rs. 40
Rebate for large Sum assured	Rs. 2 per 1000
Rebate for half-yearly payment	1.5%

Solution :

Premium Calculation	**Rs.**
Tabular Premium/1000	40.00
Less Rebate for large Sum Assured	−2.00
Less Rebate for half-yearly payment (1.5% of Rs. 40)	−0.60
Total	= 37.40

Premium = Rs. 37.40 × 50000/1000 = Rs. 1870

Half-yearly Installment Rs. 1870/2 = Rs. 935

(iii) A person at the age of 25 years takes a insurance policy of sum assured Rs. 50,000 for 30 years term. Calculate premium for quarterly payment assuming the following details:

Tabular Premium/1000	Rs. 40
Rebate for large Sum assured	Rs. 2 per 1000
Rebate for quarterly payment	NIL

Solution:

Premium Calculation	**Rs.**
Tabular Premium /1000	40.00
Less Rebate for large Sum Assured	−2.00
Total	= 38.00

Premium = Rs. 38.00 × 50000/1000 = Rs. 1900

Quarterly Installment Rs. 1900/4 = Rs. 475

(iv) A person at the age of 25 years takes a insurance policy of sum assured Rs. 50,000 for 30 years term. Calculate premium for monthly payment assuming the following details:

Tabular Premium/1000	Rs. 40
Rebate for large Sum assured	Rs. 2 per 1000
Extra premium for monthly payment	5% of tabular premium

Solution:

Premium Calculation	**Rs.**
Tabular Premium /1000	40.00
Less Rebate for large Sum Assured	−2.00
Add for Monthly Mode 5% on Rs. 40	+2.00
Total	= 40.00

Premium = Rs. 40.00 × 50000/1000 = Rs. 2000

Monthly premium = Rs. 2000/12 = Rs. 167 (rounded off)

From the above examples you must have observed that if the number of installments are more then the yearly premium will be more and if the number of installments are less then the yearly premium will be less.

	No. of Installments	Each Installment (Rs.)	Yearly Premium (Rs.)
Yearly	1	1840	1840
Half Yearly	2	935	1870
Quarterly	4	475	1900
Monthly	12	167	2000

Note : The rebates and loading are always calculated on the basic tabular premium.

Example 39.4 A man at the age of 24 years takes a whole life policy (without profits) for Rs.14000. He gets a rebate of 3% if he pays the premium annually. Find the amount of premium he has to pay if he chooses to pay the premium annually.

Solution : The tabular rate of premium = Rs.12.60

(See table 1of **Annexure- A**, in the row of 24 years)

Rebate for mode of payment = 3% of Rs. 12.60

= Rs. 0.38

∴ Premium to be paid/1000 = Rs. (12.60 – 0.38)

= Rs. 12.22

This is because there is no other adjustment or rebate.

∴ Rs. 12.22 is to be paid for a policy of Rs. 1000

∴ Premium for a policy of Rs. 14000 = Rs. $\left(\frac{12.22}{1000} \times 14000\right)$

= Rs. 171.08

annual Premium payable = Rs. 171 (after rounding off)

Example 39.5 Sohan takes a whole life policy (without profits) at the age of 28 years for Rs. 40000. If the tabular premium for half yearly premium is Rs. 20.30 ,find the amount for half yearly premium which Sohan has to pay.

Solution : Tabular premium = Rs. 20.30

Mode of payment = Half yearly

∴ Rebate for mode of payment = 1.5% of Rs. 20.30

= Rs. 0.30

∴ Balance = Rs. (20.30 – 0.30)

= Rs. 20

Rebate for large sum assured = Rs. 1.00

(because the sum assured is between Rs. 25000 and Rs. 49999)

$\therefore$ Annual Premium to be paid = Rs. (20 − 1) = Rs. 19 per thousand

$\therefore$ Annual premium to be paid = Rs. $\left(\frac{19}{1000} \times 40000\right)$

= Rs. 760

$\therefore$ Semi annual payment = Rs. $\left(\frac{760}{2}\right)$ = Rs. 380

Thus, Sohan has to pay Rs. 380 every half-yearly towards his premium.

Example 39.6 Calculate the annual premium for a whole life policy (with profits) for Rs. 85,000 taken at the age of 40 years, assuming that the mode of payment is half yearly, and the policy covers the risk of accident.

Solution :

Tabular Annual premium(See table -2) = Rs. 37.00

Rebate for mode of payment = 1.5% of Rs 37

= Rs 0.55

$\therefore$ Balance is Rs. (37 − 0.55) = Rs. 36.45

Adjustment for sum assured = Rs. 2.00

$\therefore$ Annual premium per Rs. 1000 = Rs. (36.45 − 2.00)

= Rs. 34.45

Money to be paid towards accident benefit per Rs. 1000 = Re 1

$\therefore$ Annual premium per Rs. 1000 = Rs. 35.45

$\therefore$ Annual premium to be paid = Rs. $\left(\frac{35.45}{1000} \times 85000\right)$

= Rs. 3013.25

$\therefore$ Half-yearly premium to be paid = Rs. $\left(\frac{3013.25}{2}\right)$ = Rs. 1506.62

= Rs. 1507 (after rounded off)

Thus, premium to be paid half yearly is Rs. 1507

Example 39.7 Calculate the premium for an Endowment Insurance Policy (with profits) for Rs 50,000 taken at the age of 30 years for a term of 35 years. Assume that the mode of payment is monthly under the salary saving scheme (SSS) and the policy also covers the risk of accident.

Solution: You have to see table 4 of **Annexure-A** to find the annual tabular premium.

Annual tabular premium per thousand = Rs. 30.40

Rebate for sum assured = Rs. 2

Rebate for monthly payment = Nil

$\therefore$ Annual premium per thousand = Rs 28.40

Payment for accident benefit = Re 1 per thousand

$\therefore$ Annual premium payable per thousand = Rs. (28.40 + 1)

= Rs. 29.40

$\therefore$ Total annual premium $= Rs. \left(\frac{29.40}{1000} \times 50000\right) = Rs. 1470$

$\therefore$ Monthly premium $= Rs. \left(\frac{1470}{12}\right) = Rs. 122.5$

= Rs. 122 (rounded off)

Thus, the monthly premium payable is Rs. 122.

Example 39.8 Calculate the annual premium for an Endowment insurance policy (with profits) of Rs. 40000 taken at the age of 25 years for a term of 20 years, premium paid annually and the policy is without accident benefits.

Solution : Tabular Premium = Rs. 50.80

Rebate for mode of payment = 3% of Rs. 50.80

= Rs. 1.52

$\therefore$ Balance = Rs. 49.28

Rebate for sum assured = Re. 1

$\therefore$ Annual premium per thousand = Rs. 48.28

$\therefore$ Total premium to be paid $= Rs. \left(\frac{48.28}{1000} \times 40000\right) = Rs. 1931.20$

Thus, the annual premium payable is Rs. 1931.

Example 39.9 Jitender is 30 years old and wants to purchase an Endowment insurance policy (with profits) for Rs. 90000 for a term of 25 years. Find the premium he has to pay if he pays premium half yearly and the policy is with accident benefits.

Solution : Tabular annual premium = Rs. 41.05

Rebate for the mode of payment = 1.5% of Rs. 41.05

= Rs. 0.62

$\therefore$ Balance = Rs. 40.43

Rebate for sum assured = Rs. 2 per thousand

$\therefore$ Balance to be paid per thousand = Rs. 38.43

Amounts towards accident benefit = Re 1 per thousand

$\therefore$ Annual premium to be paid per thousand = Rs. 39.43

Annual premium to be paid by Jitender $= Rs. \left(\frac{39.43}{1000} \times 90000\right) = Rs. 3548.70$

Half-yearly premium $= Rs. \left(\frac{3548.70}{2}\right) = Rs. 1774.35$

Thus, Jitender has to pay Rs. 1774 as premium half yearly.

CHECK YOUR PROGRESS 39.1

1. A man at the age of 30 years takes a whole life Policy (without profits) for Rs. 24,000. The rates of annual tabular premium being Rs. 14.95 per thousand. The corporation allows a rebate of 3% of the tabular premium if the premium is paid annually. Find the premium if it is paid annually.
2. Madhu at the age of 35 years takes a whole life policy (without profits) for Rs. 48,000 Find the amount of premium Madhu has to pay, if she chooses to pay premium half yearly.
3. A man at the age of 26 years purchases a whole life policy (with profits) for Rs.95,000. Calculate premium, assuming that the policy covers the risk of accident and is paid half yearly.
4. Calculate premium for a policy of Rs. 1,00,000 endowment insurance (with profits), if a man aged 29 years wants it for a term of 25 years. Assume that the man pays through salary saving scheme and the policy also covers the risk of accident.
5. Renuka takes an Endowment policy (with profits) for Rs. 2,00,000 for 25 years at the age of 35 years. Calculate the premium she has to pay annually if she wants to cover the risk of accident also.

39.12 PAYMENT OF BONUS

Financial soundness of a company is determined by comparing all assets with all liabilities. In a valuation of a life Insurance Company, the liabilities pertaining to life Insurance policies are worked out. Other liabilities, like outstanding capital etc. are already determined clearly and don't have to be estimated or assessed every time. The policies liabilities as determined by the valuation have to be compared with all the assets less what is earmarked for other liabilities, which are known. The net figure of assets is equal to what appears as "life fund" on the liabilities side and is the fund set aside for meeting the claim of policyholders. There is surplus if the actual life fund exceeds the liabilities shown by the valuation, which means that the fund set aside for policyholders is more than the need. If the fund is less there is deficit.

If a surplus is shown in a valuation, it has to be distributed amongst the policyholder. Any bonus distribution system should be equitable to existing and new policyholders, simple to operate, easy to understand and flexible. Following are the main systems of distribution of bonus:

1. Reversionary Bonus

Under this system, bonus is given as uniform percentage additions to the basic sum assured and is payable with the sum assured. It is called Simple Reversionary Bonus. If the bonus is calculated as a percentage of the basic sum assured plus any existing bonus previously declared it is known as Compound Reversionary Bonus.

2. Interim Bonus

Generally the bonus vests on policies that are in force on the date of valuation. Policies resulting into claims by death or maturity subsequent to the policy year containing the date of valuation will not have any bonus that year.

Therefore, interim bonus may be declared to be paid along in such claims. This is with a view to facilitate settlement of claims that may arise before the next valuation is completed and avoid reopening of all these cases at a later date.

3. Terminal Bonus

This benefit is payable on policies which are in force for the full sum assured for a minimum period of 15 years before resulting into claim by death or maturity. This bonus is in addition to the reversionary or interim bonus, if any. Terminal Bonus is not payable for paid up policies, surrendered, or discounted policies. The bonus is always paid on Sum Assured.

39.13 PAID-UP VALUE AND SURRENDER VALUE

Under Section 113 of Insurance Act, 1938, the policy of life Insurance under which the whole of benefit becomes payable on a contingency, which is bound to happen, shall if all premiums have been paid for at least 3 consecutive years, acquire a guaranteed surrender value to which shall be added the surrender value of any subsisting bonus already attached to the policy.

Further Section 113 (2) provides that a policy which has acquired surrender value shall not lapse by reason of non-payment of further premiums but shall be kept alive to the extent of paid-up sum assured and such paid-up sum shall include in full all subsisting reversionary bonuses that have already attached to the policy. Such amount shall not be (Excluding attached bonuses) less than the amount bearing to the total period for which premiums have already been paid bears to the maximum period for which premiums were originally payable.

The paid up value is calculated as under:

$$\textbf{Paid Up Value(PV)} = \frac{\text{Number of Instalments Paid} \times \text{Sum Assured}}{\text{Total No. of Instalments Payable}} + \text{Bonus (if any)}$$

$$\textbf{Surrender Value (SV)} = \text{Paid-up Value} \times \text{Surrender Value Factor}/100$$

The Surrender Value factor depends upon the following :

- Rate of interest earned by the Insurer.
- The payment made in advance for number of years.

The table showing the **Surrender Value Factor** is given in **Annexure-'B'**.

To illustrate the paid up and surrender value the examples are given below:

Example 39.10 A person at the age of 35 years takes an insurance policy for a term of 20 years on 01-04-1990 for Rs. 1,00,000. The last premium paid is on 01-04-2001 Calculate the paid-up value and surrender value given that the Surrender Value Factor is 60% and mode of payment is **(a) yearly (b) half yearly (c) quarterly (d) monthly.**

Solution:

(a) If mode of payment is yearly:

Number of instalments paid = (1/4/2001 – 1/4/90) + 1 = 12

(Because the instalment on 1/4/90 is also paid)

Total instalments payable = 20

$$\text{Paid up value} = Rs.\ \frac{12}{20} \times 1{,}00{,}000 = Rs.\ 60{,}000$$

Surrender value $= Rs.\ 60{,}000 \times 60\% = Rs.\ 36{,}000$

(b) If mode of payment is half-yearly:

Number of instalments paid $= (1/4/2001 - 1/4/90) \times 2 + 1 = 23$

(Multiplied by 2 because of half-yearly mode and added 1 because the instalment on 1/4/90 is also paid)

Total instalments payable $= 20 \times 2 = 40$

Paid up value $= Rs.\ \frac{23}{40} \times 1{,}00{,}000 = Rs.\ 57{,}500$

Surrender value $= Rs.\ 57{,}500 \times 60\% = Rs.\ 34{,}500$

(c) If mode of payment is quarterly:

Number of instalments paid $= (1/4/2001 - 1/4/90) \times 4 + 1 = 45$

(Multiplied by 4 because of quarterly mode and added 1 because the instalment on 1/4/90 is also paid)

Total instalments payable $= 20 \times 4 = 80$

Paid up value $= Rs.\ \frac{45}{80} \times 1{,}00{,}000 = Rs.\ 56{,}250$

Surrender value $= Rs.\ 56{,}250 \times 60\% = Rs.\ 33{,}750$

(d) If mode of payment is monthly:

Number of instalments paid $= (1/4/2001 - 1/4/90) \times 12 + 1 = 133$

(Multiplied by 12 because of monthly mode and added 1 because the instalment on 1/4/90 is also paid)

Total instalments payable $= 20 \times 12 = 240$

Paid up value $= Rs.\ \frac{133}{240} \times 1{,}00{,}000 = Rs.\ 55{,}416$

Surrender value $= Rs.\ 55{,}416 \times 60\% = Rs.\ 33{,}250$

Notes

39.13.1 Guaranteed Surrender Value

The minimum surrender values is equal to 30% of the total amount of premiums paid excluding first year premium and additional premiums for additional benefits. In addition, cash value of existing bonus will also be allowed subject to surrender value factor. This is known as "Guaranteed Surrender Value". As per the provisions of Insurance Act, the surrender value formula has to be mentioned in policy documents.

Minimum (guaranteed) surrender value allowable = 30% of the total premiums paid (a) excluding the First year Premium (b) all extra Premiums and (c) additional premiums for D.A.B. **The Surrender of policy would mean cancellation of the contract.**

Example 39.11 A person at the age of 30 years takes a endowment policy on 14-06-1989 for Rs. 25,000 for 25 years term. Calculate the paid-up value if the last premium paid is on 14.06.1997 and the mode of payment is quarterly.

Solution: Policy in force up to : = 14/06/1997

Total number of premiums paid $= (14/6/1997 - 14/6/89) \times 4 + 1 = 33$

Total number of premiums payable

(Term × number of premiums in a year depending on mode) $= 25 \times 4 = 100$

Paid-up value $= Rs.\ (33 \times 25000)/100 = Rs.\ 8250$

Example 39.12 In Example 39.11, what will be the paid-up value if the consolidated Reversionary Bonus declared by the insurer from March, 1990 to March, 1996 is Rs. 410 per thousand S.A. bonus declared for the year ending March 1997 is @ Rs. 70 per thousand.

Solution: Paid-up value (as derived from Example 39.11) = Rs.8,250

Total paid-up value = PV + Bonuses

Bonus for valuations 3/1990 to 3/1996 $= Rs.\ (410 \times 25) = Rs.\ 10250$

(@ Rs. 410 per thousand sum assured)

Bonus for valuation of 3/1997 @ Rs. 70 per thousand $= Rs.\ (70 \times 25) = Rs.\ 1750$

Total PV (including bonus) = Rs. (8250 + 10250 + 1750)

= Rs. 20250

39.14 MEDICLAIM INSURANCE

Mediclaim Insurance is meant to cover the medical expanses incurred by the insured during the period of insurance under the policy.

1. Salient Features of the Policy

(a) The Policy covers reimbursement of Hospitalization and/or Domiciliary Hospitalization expenses only for illness/disease contracted or injury sustained by the Insured Person.

(b) In the event of any claim becoming admissible under this scheme, the Company will pay to the Insured Person the amount of such expenses as would fall under different heads mentioned below as are reasonably and necessarily incurred in respect thereof anywhere in India by or on behalf of such Insured Person, but not exceeding Sum Insured for that person as stated in the Schedule in any one period of Insurance.

(i) Room, Boarding Expenses as provided by the Hospital/Nursing Home.

(ii) Nursing Expenses

(iii) Surgeon, Anesthetist, Medical Practitioner, Consultants, Specialist fees.

(iv) Anesthesia, Blood, Oxygen, Operation Theater Charges, Surgical Appliances, Medicines & Drugs, Diagnostic Materials and X-Ray, Dialysis, Chemotherapy, Radiotherapy, Cost of Pacemaker, Artificial Limbs and Cost of Organs and similar expenses.

Note : Company's Liability in respect of all claims admitted during the period of insurance shall not exceed the Sum Insured per person mentioned in the Schedule.

(c) "Surgical Operation" means manual and/or operative procedures for correction of deformities and defects, repair of injuries, diagnosis and cure of diseases, relief of suffering and prolongation of life.

(d) Expenses on Hospitalization for minimum period of 24 hours are admissible. However, this time limit is not applied to specific treatment i.e. Dialysis, Chemotherapy, Radiotherapy, Eye-Surgery, Lithotripsy (Kidney stone removal), Tonsillectomy, Dog bite, D&C taken in the Hospital/Nursing Home and the insured is discharged on the same day, the treatment will be considered to be taken under Hospitalization Benefit.

39.14.1 Pre-Hospitalization

Relevant medical expenses incurred during period upto 30 days prior to hospitalization on disease/injury sustained will be considered as part of claim.

39.14.2 Post-Hospitalization

Relevant medical expenses incurred during period upto 60 days after hospitalization on disease/ illness/injury sustained will be considered as part of claim.

39.15 AGE LIMIT

This insurance is available to persons between the age of 5 years and 75 years. For fresh proposals above 60, acceptance subject to satisfactory medical examination at proposer's cost. Children between the age of 3 months and 5 years of age can be covered provided one or both parents are covered concurrently. Where the Insured Person is over 75 years of age as on Renewal date. Renewal of Mediclaim Policy should be subject to the following:

(a) There should not be any break in renewal.

(b) Premium chargeable upto 80 years should be the same as that applicable for the age group 71 to 75 years. For persons over 80 years of age, following loading should be carried out:
  (i) Upto 85 years : 15%
  (ii) Over 85 years : 25%

(c) Renewal of Policies over 75 years should always be on expiring Sum Insured Basis.

(d) While intention should be to allow continuation of such policies subject to "No Break" all renewal cases of over 80 years should be approved at the Manager's level in the Regional Office once in two years but well before the Renewal date by judiciously reviewing the claims experience under the Policy vis-à-vis other relevant factors.

Note : the above provision would apply only in case of policies issued under Standard Mediclaim Scheme.

39.16 SUM INSURED

Rs.15,000 (Minimum) and Rs. 5 lakhs (Maximum) for one year. **For detail mediclaim** schedule, you may refer to **Annexure -'C'.**

39.17 FAMILY DISCOUNT

A discount of 10% in the total premium will be allowed comprising the insured and any one or more of the following:

(i) Spouse

(ii) Dependent Children (i.e. legitimate or legally adopted children)

(iii) Dependent parents

39.18 NOTICE OF CLAIM

Preliminary notice of claim with particulars relating to policy numbers, Name of the Insured Person in respect of whom claim is made. Nature of illness/injury and Name and Address of the attending Medical Practitioner/Hospital/Nursing Home should be given by the Insured Person to the Company within seven days from the date of Hospitalization/Domiciliary Hospitalization. Final Claim along with receipted Bills/Cash Memos, Claim Form and list of documents as listed in the claim form etc. should be submitted to the Company within 30 days of discharge from the Hospital.

39.19 COST OF HEALTH CHECK-UP

The Insured shall be entitled for reimbursement of cost of medical checkup once at the end of every four underwriting years provided there are no claims reported during the block. The cost so reimbursement shall not exceed the amount equal to 1% of the average Sum Insured during the block of four claims free underwriting years.

39.20 PERIOD OF INSURANCE

Policy is issued for one year. No Short Period Policy.

Note: When an individual having Mediclaim Policy and traveling abroad with OMP the Mediclaim Policy shall stand suspended for the period the insured is covered under OMP. Therefore, the Mediclaim Policy shall stand extended for the same period beyond the expiry date, thus no adjustment/refund of premium would be involved.

Example 39.13 Ram Avtar takes a Mediclaim Insurance Policy for himself for Rs. 2 lakhs. His age is 61 years. Consult the table given below and find out the annual premium he has to pay. If 12% is the service charge on the premium, calculate the total amount he has to pay for this Insurance. **(See Annexure -'C')**

Sum Insured (Overall Liabilities in Rs.)	Amount of liability for Domiciliary Hospitalization	Upto 35 years	36-45 years	46-55 years	56-65 years	66-70 years	71-75 years	76-80 years
195000	34250	2414	2624	3811	4354	4892	5340	6918
200000	35000	2469	2683	3900	4458	5010	5471	7097
205000	35500	2518	2736	3982	4555	5120	5595	7269

Solution : From the given table, the annual premium = Rs. 4458

$$\text{Service charge} = Rs.[4458 \times \frac{12}{100}] = \text{Rs.}534.96 \approx \text{Rs.}535$$

Total amount to be paid = Rs.(4458 + 535) = Rs. 4993

Example 39.14 John wants to take a Mediclaim Insurance Policy for self, his wife and daughter for Rs.2 lakhs for self, 2.05 lakhs for his wife and 1.95 lakhs for his daughter. Their ages are 63 years, 55 years and 25 years respectively. 10% discount on the premium is allowed for family insurance. If 10% service charge is levied on the net premium, calculate the total amount to paid for the Mediclaim Insurance of the family consulting the following premium table:

Sum Insured (Overall Liabilities in Rs.)	Amount of liability for Domiciliary Hospitalization	Upto 35 years	36-45 years	46-55 years	56-65 years	66-70 years	71-75 years	76-80 years
195000	34250	2414	2624	3811	4354	4892	5340	6918
200000	35000	2469	2683	3900	4458	5010	5471	7097
205000	35500	2518	2736	3982	4555	5120	5595	7269

Solution : Premium for self = Rs.4458

Premium for wife = Rs. 3982

Premium for daughter = Rs.2414

Total Premium = Rs.10854

Discount (10%) = Rs.1085.40 ≈ Rs. 1085

∴ Net Premium = Rs.(10854 − 1085) = Rs. 9769

Service Charge = Rs. 976.9 ≈ Rs. 977

∴ Total Amount to be paid = Rs. [9769 + 977] = Rs. 10746.

39.21 GENERAL INSURANCE

The Insurance of goods, property, vehicles etc. is called General Insurance. The General Insurance Company (GIC) was formed in 1973. There are four companies under GIC.

They are:

(1) National Insurance Company Limited

(2) New India Insurance Company Limited

(3) Oriental Insurance Company Limited

(4) United Insurance Company Limited

All these companies have the same rate of premium (but the names of the schemes may be different).

The various schemes of the GIC cover

(1) Movable and Immovable Property Insurance

(2) Vehicle Insurance

(3) Goods in Transit Insurance

Let us study something about each one of them.

1. Movable and Immovable Property Insurance

Movable and Immovable Property belonging to an individual or an organization can be insured against fire, theft, natural calamities, riots etc.

2. Vehicle Insurance

There are two types of insurance for vehicle.

(i) **Act Insurance or the Third party Insurance :** This is compulsory for all vehicles under the Motor Vehicle Act. If a vehicle is insured under "Act Insurance" only, then the damage caused to another person or his property (Including Vehicle) is payable by the company in case of an accident i.e. the person who suffers the loss is compensated and not the insured. The insured doesn't get any compensation.

The rates of annual premium at present are:

(a) Rs. 160 for cars;(b) Rs. 40 for two-wheeler scooters; (c) Rs. 30 for Mopeds

(ii) **Comprehensive Insurance :** Under this scheme, the person whose vehicle is assured also gets compensation, in addition to the money paid to the third party.

Thus, the insured also gets a cover for the damage or loss suffered by him (or her) or his/her vehicle.

No Claim Bonus: If no claim is made during the year of comprehensive insurance, the company allows a rebate to the insured (i.e. owner of the vehicle) in the premium to be paid in the successive year. The rate of rebate continues to increase year after year if no claim is made on the policy. This is called "No Claim Bonus".

Note: 'No Claim Bonus' is not given on 'Act Insurance'.

The present rates of 'No Claim Bonus' are as under"

Year	**Car**	**Scooter**
First Year	15%	15%
Second Year	30%	25%
Third Year	45%	30%
Fourth Year	60%	40%
Fifth Year and after	60%	60%

Thus, the maximum **'No Claim Bonus'** for cars is 60% after 4 consecutive years of not claiming the policy and for two wheelers it is 60% after 5 consecutive years of not claiming the policy.

'No Claim Bonus' is a sort of reward for not claiming any damages by the insured and not for the vehicle. Thus, an insured who manages his (or her) car without any claim for 5 years, pay only 40% of the basic premium.

(3) **Goods in transit insurance :** When goods are sent from one place to another, there is a possibility of loss/damage occurring in transit due to accident, strike, riots etc. The mode of transit could be road, rail, sea or air. To cover such risk, there are many policies with different rates of premium.

Difference between Life Insurance and General Insurance

Life Insurance	**General Insurance**
1.Amount assured becomes payable on death or maturity.	Actual financial loss suffered becomes payable subject to the limit of sum assured.
2.It is for whole life or for a specified period	Usually for a period of one year
3.It promotes savings alongwith financial protection.	It provides financial protection only.

We will now take some examples.

Example 39.15 A man insured his house for Rs. 4,20,000 against fire and other calamities at the rate of 1% premium. What annual premium he has to pay?

Solution : Value of the house = Rs. 420000

Rate of premium = Rs. 10 per thousand

$\therefore$ Annual premium $= Rs.\left(\dfrac{420000 \times 10}{1000}\right)$

= Rs. 4200

Thus, the annual premium to be paid is = Rs. 4200

Example 39.16 If a car costs Rs. 2,20,000, then what will be the comprehensive insurance of the car if the tabular premium charged is Rs. 4113 for Rs. 1,30,000 and 2.95% of the excess amount and the act insurance is Rs. 160.

Solution : Tabular premium = Rs. 4113

Excess amount $= Rs.\ (2,20,000 - 1,30,000) = Rs.\ 90,000$

$\therefore$ The premium on excess amount $= Rs.\left(\dfrac{2.95}{100} \times 90000\right) = Rs.\ 2655$

$\therefore$ Premium = Rs. (4113 + 2655) = Rs. 6768

Act Insurance = Rs. 160

$\therefore$ Premium to be paid = Rs. (6768 + 160)

= Rs. 6928

Thus, premium to be paid = Rs. 6928

Note : If the owner of the car is allowed No Claim Bonus, it will be calculated on Rs. 6768 and then Act insurance will be added.

Example 39.17 The comprehensive insurance charges for a two-wheeler scooter costing Rs. 42000 are Rs. 432. If a no claim bonus @ 30% is allowed and the 'Act Insurance' is Rs. 40, find the premium to be paid for the renewal of the policy next year.

Solution : Tabular Premium = Rs. 432

No Claim Bonus = 30% of Rs. 432

$= Rs.\left(\dfrac{30}{100} \times 432\right) = Rs.\ 129.6$

Premium = Rs. (432 – 129.6)

= Rs. 302.40

Act Insurance = Rs. 40

$\therefore$ Premium to be paid = Rs. (302.40 + 40)

= Rs. 342.40 = Rs. 342

Thus, the premium to be paid is Rs. 342.

Example 39.18 A man is transferred from Delhi to Mumbai and he wants to insure his house hold goods against accident, riots and other perils. The present rate of insurance is 95 paise per hundred rupees. If he insures his household goods for Rs. 90,000 what premium the man will pay for his transit insurance?

Solution : Present rate 95 paise per hundred i.e. Rs. 9.5 per thousand

Total Premium for transit $= \text{Rs. } 90 \times 9.5 = Rs.\ 855$

Example 39.19 The owner of the car is entitled for no claim bonus of 60% after 4 years. If the owner purchases a new car after 4 years for Rs. 2,25,000 what premium the owner will pay for his comprehensive insurance? The tabular value for Rs. 1,30,000 is Rs. 4113 + 2.95% of the excess amount if amount (cost) is more than Rs. 130000.

Solution : Tabular premium for Rs. 1,30,000 = Rs.4113

For extra 95,000 premium = 2.95% of 95,000 = Rs. 2802.5 ≈ Rs. 2802

Premium for Rs. 2,25,000 = Rs.(4113 + 2802) = Rs. 6915

No claim bonus = 60% of Rs. 6915 = $Rs.\left(\frac{60}{100} \times 6915\right) = Rs.\ 4149$

Balance = Rs. (6915 − 4149) = Rs. 2766

Adding Act Insurance (Third party) = Rs. 160

Total annual premium = Rs. (2766 + 160) = Rs. 2926

Thus, premium to be paid is Rs. 2926.

CHECK YOUR PROGRESS 39.2

1. A man insured his house and belongings against fire and other natural calamities for Rs. 1,60,000 at the rate of Re. 1 per thousand. What is the annual premium he has to pay?

2. Ms. Mehta was transferred from Mumbai to Chennai. She insured her belongings for Rs. 4,00,000 at the rate of 1% premium. What is the amount of premium?

3. A comprehensive insurance of a car was taken by Robert. If the cost of his car is Rs. 130000 and the tabular value of the premium is Rs. 4113, what is the annual premium if he has had 7 years of accident free driving?

   Assume act insurance = Rs. 160 (maximum no claim bonus allowed = 60%)

4. What is the annual premium for a comprehensive policy if the Act Insurance is Rs.160, cost of the car is Rs. 2,05,000 and 'No Claim Bonus' not admissible on his insurance. Assume that the rate of premium is Rs. 4113 for Rs. 130000 and 2.95% for the extra amount.

5. Arun wants to renew the 'comprehensive insurance' of his car costing Rs. 2,95,000 after having a 6 year period of accident free driving. How much more amount would he

pay in getting the insurance renewed over getting a new 'Act Insurance Policy'? (Tabular value of premium = Rs. 4113 for Rs. 130000 and excess @ 2.95%, Maximum 'No Claim Bonus' = 60% of the premium and Act Insurances is Rs. 160).

LET US SUM UP

- Insurance is an agreement or a contract between the insured and the Insurance Company (Insurer).
- By paying the insurance premium,the insured obtains significant relief in income tax and wealth tax.
- Adjustments are made for different mode of payments as per details given below:

Mode	Rebates
1. Yearly	3% of Tabular premium
2. Half-Yearly	1.5% of Tabular premium
3. For quarterly mode and monthly mode under Salary Saving Scheme (SSS)	No Rebate : No loading
4. For ordinary monthly mode except Salary Saving Scheme for monthly payment	Loading of 5% on Tabular Premium

- Rebates assumed for large sum assured are as follows:

Sum Assured	Rebates
1. Upto Rs. 24,999	No Rebate
2. From Rs. 25,000 to Rs. 49,999	@ Re 1 per thousand sum assured
3. From Rs. 50,000 and above	@ Rs. 2 per thousand sum assured
Extra Premium to be charged for grant of Double Accident Benefit (DAB) and Extended Permanent Disability Benefit (EPDB)	@ Re. 1 per thousand sum assured

- **Paid Up Value(PV)** $= \dfrac{\text{Number of Instalments Paid} \times \text{Sum Assured}}{\text{Total No. of Instalments Payable}} + \text{Bonus (if any)}$
- **Surrender Value (SV)** $= \text{Paid-up Value} \times \text{Surrender Value Factor}/100$
- The rates of "No claim Bonus" are given below:

Year	Car	Scooter
First Year	15%	15%
Second Year	30%	25%
Third Year	45%	30%
Fourth Year	60%	40%
Fifth Year and after	60%	60%

SUPPORTIVE WEB SITES

- http://www.licindia.com
- http://www.nationalinsuranceindia.com
- http://www.irdaindia.org
- http://www.insuremarket.com

TERMINAL EXERCISE

1. Reena takes a whole life policy (without profits) at the age of 30 years for Rs. 50,000. If a rebate of 3% is allowed for mode of payment and Rs. 2 per thousand for the sum assured, find the annual premium she has to pay.
2. Rashmi takes a whole life policy with risk coverage at the age of 35 years for Rs. 2,00,000. Find the annual premium that she has to pay if a rebate of 3% of tabular premium for the mode of payment and Rs. 2 per thousand for the sum assured are allowed.
3. Ram Prasad takes an Endowment policy (without profits) for Rs. 4,00,000 at the age of 40 years for 10 years, which covers accident risk also. Find the premium he has to pay annually if 3% rebate is given for annual payment of premium.
4. Mr. Mohan Kumar insured his house for Rs. 2,80,000 against fire and other risks. If the rate of premium is 2%, per annum find the annual premium.
5. Nillu renewed her 'Comprehensive Insurance' for her car costing Rs. 2,50,000 after having 2 years of accident free driving. If the rate of premium is Rs. 4773 for Rs. 150000 and 2.95% for the excess amount, find the premium she has to pay.
6. Sheetal takes a whole life policy (with profits) at the age of 40 for Rs. 2,00,000 and pays the premium annually. If a rebate of 3% is allowed for this mode of payment and Rs. 2 per thousand for the sum assured, find the annual premium she has to pay.
7. Rooma takes a whole life policy (without profits) at the age of 50 years for Rs. 3,00,000. Find the annual premium she has to pay if 3% rebate is allowed for this mode of payment and rebate @ Rs 2 per thousand is allowed for the sum assured, and the policy covers the risk of accident also (Re 1 per Rs 1000 to be paid towards premium for this benefit).
8. Garima takes an Endowment insurance policy (without profits) of Rs. 1,00,000 at the age of 22 years for a term of 25 years. Find the annual premium that she has to pay if the policy is with accident benefits.
9. Dr. Saurav wants to take a Mediclaim Insurance Policy for self, his wife and daughter for Rs.2.7 lakhs for self, 3.5 lakhs for his wife and 3 lakhs for his daughter. Their ages are 40 years, 38 years and 6 years respectively. 10% discount on the premium is allowed for family insurance. If 10% service charge is levied on the net premium, calculate the total amount to be paid for the Mediclaim Insurance of the family consulting the following premium table:

Sum Insured (Overall Liabilities in Rs.)	Amount of liability for Domiciliary Hospitalization	Upto 35 years	36-45 years	46-55 years	56-65 years	66-70 years	71-75 years	76-80 years
270000	42000	3151	3425	5057	5809	6553	7207	9508
300000	45000	3444	3743	5553	6388	7214	7951	10542
350000	50000	3870	4207	6311	7283	8248	9122	12195

ANSWERS

CHECK YOUR PROGRESS 39.1

1. Rs. 348 2. Rs. 400 3. Rs. 1054
4. Rs. 332 5. Rs. 8094

CHECK YOUR PROGRESS 39.2

1. Rs. 160 2. Rs. 4000 3. Rs. 1085
4. Rs. 6485 5. Rs. 3752

TERMINAL EXERCISE

1. Rs. 625 2. Rs. 5634 3. Rs. 32,736
4. Rs. 5600 5. Rs. 4248 6. Rs. 6778
7. Rs. 10.350 8. Rs. 2200 9. Rs. 10,965

Annexure-A

THE WHOLE LIFE POLICY

Annual Premiums payable untill death for the Insurance of Rs. 1000 to be paid at death only.

Table -1
(Without Profits)

Age nearer Birthday Years	Annual Premiums Rs. P.	Age nearer Birthday Years	Annual Premiums Rs.P.
15	10.75	39	21.25
16	10.90	40	22.25
17	11.05	41	23.30
18	11.20	42	24.45
19	11.40	43	25.65
20	11.60	44	26.95
21	11.80	45	28.35
22	12.05	46	29.80
23	12.35	47	31.35
24	12.60	48	33.00
25	12.95	49	34.75
26	13.30	50	36.60
27	13.65	51	38.60
28	14.05	52	40.75
29	14.50	53	43.05
30	14.95	54	45.45
31	15.50	55	48.05
32	16.00	56	50.80
33	16.60	57	53.75
34	17.85	58	56.90
35	17.95	59	60.25
36	18.65	60	63.80
37	19.45		
38	20.30		

Table -2
(With Profits)

Age nearer Birthday Years	Annual Premiums Rs. P.	Age nearer Birthday Years	Annual Premiums Rs.P.
15	17.95	39	35.70
16	18.35	40	37.00
17	18.80	41	38.35
18	19.20	42	39.80
19	19.65	43	41.30
20	20.15	44	42.90
21	20.65	45	44.60
22	21.15	46	46.30
23	21.75	47	48.05
24	22.30	48	49.90
25	22.90	49	51.85
26	23.55	50	53.95
27	24.25	51	56.15
28	24.95	52	58.50
29	25.70	53	60.95
30	26.50	54	63.60
31	27.35	55	66.35
32	28.20	56	69.30
33	29.10	57	72.40
34	30.05	58	75.70
35	31.10	59	79.15
36	32.15	60	82.85
37	33.25		
38	34.45		

The Endowment Insurance Policy (Without Profits)

Table -3

Annual Premiums for the Insurance of Rs. 1000 to be paid at the end of specified term of years or at death, if it should occur sooner

	Endownment Term							
Age nearer Birthday Years	5 years Rs. P.	6 years Rs. P.	7 years Rs. P.	8 years Rs. P.	9 years Rs. P.	10 years Rs. P.	11 years Rs. P.	Age nearer Birthday Years
15	188.90	152.00	126.00	106.80	92.00	83.40	73.45	15
16	188.90	152.00	126.00	106.80	92.00	83.40	73.45	16
17	188.90	152.00	126.00	106.80	92.00	83.40	73.45	17
18	188.90	152.00	126.00	106.80	92.00	83.40	73.45	18
19	188.90	152.00	126.00	106.80	92.00	83.40	73.45	19
20	188.90	152.00	126.00	106.80	92.00	83.40	73.45	20
21	188.90	152.00	126.00	106.80	92.00	83.40	73.45	21
22	188.90	152.00	126.00	106.80	92.00	83.40	73.45	22
23	188.90	152.00	126.00	106.80	92.00	83.40	73.45	23
24	188.90	152.00	126.00	106.80	92.00	83.40	73.45	24
25	188.90	152.00	126.00	106.80	92.00	83.40	73.45	25
26	188.90	152.00	126.00	106.80	92.00	83.40	73.50	26
27	188.90	152.00	126.00	106.80	92.05	83.45	73.55	27
28	188.90	152.00	126.05	106.85	92.10	83.50	73.60	28
29	188.95	152.05	126.10	106.90	92.15	83.60	73.65	29
30	189.00	152.10	126.15	106.95	92.25	83.65	73.75	30
31	189.05	152.15	126.20	107.05	92.30	83.75	73.85	31
32	189.10	152.25	126.30	107.10	92.40	83.85	73.95	32
33	189.20	152.30	126.40	107.20	92.50	83.95	74.05	33
34	189.25	152.40	126.45	107.30	92.60	84.05	74.20	34
35	189.35	152.50	126.55	107.40	92.70	84.20	74.35	35
36	189.45	152.60	126.70	107.55	92.85	84.40	74.55	36
37	189.55	152.70	126.80	107.70	93.05	84.60	74.75	37
38	189.70	152.85	127.00	107.90	93.25	84.80	75.00	38
39	189.85	153.05	127.20	108.10	93.50	85.10	75.30	39
40	190.05	153.30	127.45	108.35	93.75	85.40	75.65	40
41	190.30	153.55	127.70	108.65	94.05	85.75	76.00	41
42	190.55	153.80	128.00	109.00	94.40	86.15	76.40	42
43	190.85	154.15	128.35	109.35	94.80	86.60	76.90	43
44	191.15	154.45	128.75	109.75	95.25	87.05	77.40	44
45	191.50	154.85	129.15	110.20	95.70	87.60	77.95	45
46	191.90	155.25	129.60	110.65	96.20	88.15	78.55	46
47	192.30	155.75	130.10	111.20	96.80	88.80	79.20	47
48	192.75	156.25	130.65	111.80	97.40	89.50	79.95	48
49	193.30	156.80	131.25	112.45	98.10	90.30	80.80	49
50	193.85	157.40	131.90	113.15	98.85	91.15	81.65	50
51	194.45	158.10	132.60	113.90	99.65	92.05	82.65	51
52	195.15	158.80	133.40	114.75	100.55	93.05	83.70	52
53	195.85	159.60	134.30	115.70	101.55	94.20	84.90	53
54	196.70	160.50	135.20	116.70	102.65	95.40	86.15	54
55	197.55	161.45	136.25	117.80	103.80	96.70	87.55	55
56	198.55	162.50	137.40	119.00	105.10	98.20	89.10	56
57	199.60	163.65	138.65	120.35	106.50	99.75	90.75	57
58	200.75	164.90	139.95	121.75	108.05	101.50	92.55	58
59	202.00	166.25	141.45	123.35	109.70	103.35	94.55	59
60	203.35	167.75	143.05	125.05	111.55	105.45	95.05	60

The Endowment Insurance Policy (without profits) Table-3 **contd..**

Annual Premiums for the Insurance of Rs. 1000 to be paid at the end of specified term of years or at death, if it should occur sooner

Age nearer Birthday Yrs.	ENDOWMENT TERM							Age nearer Birthday Yrs.
	12 Years Rs. P.	13 Years Rs. P.	14 Years Rs. P.	15 Years Rs. P.	16 Years Rs. P.	17 Years Rs. P.	18 Years Rs. P.	
15	65 30	58 50	52 80	48 85	44 55	40 85	37 65	15
16	65 30	58 50	52 80	48 85	44 55	40 85	37 65	16
17	65 30	58 50	52 80	48 85	44 55	40 85	37 65	17
18	65 30	58 50	52 80	48 85	44 55	40 85	37 65	18
19	65 30	58 50	52 80	48 85	44 55	40 85	37 65	19
20	65 30	58 50	52 80	48 85	44 55	40 85	37 65	20
21	65 30	58 50	52 80	48 85	44 55	40 85	37 65	21
22	65 30	58 50	52 80	48 85	44 60	40 90	37 70	22
23	65 30	58 50	52 80	48 85	44 60	40 95	37 75	23
24	65 30	58 55	52 85	48 90	44 65	41 00	37 80	24
25	65 35	58 60	52 90	48 95	44 70	41 05	37 85	25
26	65 35	58 65	52 95	49 00	44 75	41 10	37 90	26
27	65 45	58 70	53 00	49 10	44 85	41 20	38 00	27
28	65 50	58 75	53 10	49 15	44 90	41 25	38 10	28
29	65 55	58 85	53 15	49 20	45 05	41 40	38 20	29
30	65 65	58 95	53 25	49 35	45 15	41 50	38 35	30
31	65 75	59 05	53 40	49 50	45 30	41 65	38 50	31
32	65 85	59 15	53 50	49 65	45 45	41 80	38 70	32
33	66 00	59 30	53 65	49 80	45 60	42 00	38 90	33
34	66 15	59 45	53 85	50 00	45 80	42 20	39 10	34
35	66 30	59 65	54 05	50 20	46 05	42 45	39 35	35
36	66 60	59 85	54 25	50 45	46 30	42 75	39 65	36
37	66 75	60 10	54 50	50 75	46 60	43 05	40 00	37
38	67 00	60 40	54 85	51 05	46 95	43 45	40 40	38
39	67 30	60 70	55 20	51 45	47 35	43 85	40 85	39
40	67 65	61 10	55 55	51 85	47 80	44 30	41 30	40
41	68 05	61 60	56 00	52 35	48 30	44 85	41 85	41
42	68 50	61 95	56 50	52 85	48 85	45 40	42 45	42
43	69 00	62 45	57 00	53 45	49 45	46 05	43 10	43
44	69 50	63 00	57 60	54 05	50 10	46 75	43 85	44
45	70 10	63 65	58 25	54 75	50 85	47 50	44 65	45
46	70 75	64 30	58 95	55 55	51 65	48 35	45 50	46
47	71 45	65 05	59 75	56 40	52 50	49 25	46 45	47
48	72 25	65 90	60 60	57 30	53 50	50 25	47 50	48
49	73 10	66 80	61 55	58 30	54 55	51 35	48 65	49
50	74 00	67 75	62 60	59 40	55 70	52 55	49 90	50
51	75 05	68 85	63 70	60 65	56 95	53 90	51 30	51
52	76 15	70 00	64 95	61 95	58 35	55 30	52 80	52
53	77 40	71 30	66 30	63 40	59 85	56 90	—	53
54	78 75	72 70	67 75	65 00	61 50	—		54
55	80 20	74 25	69 40	66 75	—			55
56	81 80	75 95	71 15	—				56
57	83 55	77 75	—					57
58	85 45	—						58

(Contd.)

Age nearer Birthday Yrs.	ENDOWMENT TERM							Age nearer Birthday Yrs.
	19 Years Rs. P.	20 Years Rs. P.	21 Years Rs. P.	22 Years Rs. P.	23 Years Rs. P.	24 Years Rs. P.	25 Years Rs. P.	
15	34 85	32 35	30 15	28 20	26 45	24 90	23 50	15
16	34 85	32 35	30 15	28 20	26 45	24 90	23 50	16
17	34 85	32 35	30 15	28 20	26 45	24 90	23 50	17
18	34 85	32 35	30 15	28 20	26 45	24 90	23 55	18
19	34 85	32 35	30 15	28 20	26 50	24 95	23 55	19
20	34 85	32 35	30 20	28 25	26 50	24 95	23 60	20
21	34 85	32 40	30 20	28 25	26 55	25 00	23 65	21
22	34 90	32 40	30 25	28 30	26 60	25 05	23 70	22
23	34 95	32 45	30 30	28 35	26 65	25 15	23 75	23
24	35 00	32 55	30 35	28 45	26 75	25 20	23 85	24
25	35 05	32 60	30 45	28 50	26 80	25 30	23 95	25
26	35 15	32 70	30 50	28 60	26 90	25 40	24 05	26
27	35 20	32 80	30 65	28 75	27 05	25 55	24 20	27
28	35 35	32 90	30 75	28 85	27 20	25 70	24 35	28
29	35 45	33 05	30 90	29 00	27 35	25 85	24 55	29
30	35 60	33 20	31 05	29 20	27 55	26 05	24 75	30
31	35 75	33 35	31 25	29 40	27 75	26 30	25 00	31
32	35 95	33 55	31 45	29 60	27 95	26 55	25 25	32
33	36 15	33 80	31 70	29 85	28 25	26 80	25 55	33
34	36 40	34 05	31 95	30 15	28 55	27 10	25 85	34
35	36 70	34 35	32 25	30 45	28 90	27 50	26 25	35
36	37 00	34 65	32 60	30 85	29 25	27 85	26 65	36
37	37 35	35 05	33 00	31 25	29 70	28 30	27 10	37
38	37 75	35 45	33 45	31 70	30 20	28 85	27 65	38
39	38 20	35 95	33 95	32 25	30 70	29 40	28 25	39
40	38 70	36 45	34 50	32 80	31 30	30 05	28 90	40
41	39 30	37 05	35 15	33 45	32 00	30 70	29 60	41
42	39 90	37 70	35 80	34 15	32 70	31 50	30 40	42
43	40 60	38 45	36 55	34 95	33 55	32 30	31 25	43
44	41 35	39 20	37 35	35 80	34 40	33 20	32 20	44
45	42 20	40 10	38 25	36 70	35 35	34 20	33 20	45
46	43 10	41 05	39 25	37 75	36 40	35 30	—	46
47	44 10	42 05	40 35	38 85	37 55	—		47
48	45 20	43 20	41 50	40 05	—			48
49	46 40	44 45	42 80	—				49
50	47 70	45 80	—					50
51	49 10	—						51

(Contd.)

Notes

The Endowment Insurance Policy (without profits) Table-3 **contd..**

Annual Premiums for the Insurance of Rs. 1000 to be paid at the end of specified term of years or at death, if it should occur sooner

Age nearer Birthday Yrs.	ENDOWMENT TERM 26 Years Rs. P.	27 Years Rs P	28 Years Rs. P.	29 Years Rs. P.	30 Years Rs. P.	31 Years Rs. P	32 Years Rs. P.	33 Years Rs. P	Age nearer Birthday Yrs
15	22 25	21 15	20 10	19 20	18 35	17 60	16 95	16 30	15
16	22 25	21 15	20 10	19 20	18 35	17 60	16 95	16 35	16
17	22 25	21 15	20 15	19 20	18 40	17 65	16 95	16 35	17
18	22 30	21 15	20 15	19 25	18 45	17 70	17 00	16 40	18
19	22 30	21 20	20 20	19 30	18 45	17 75	17 05	16 45	19
20	22 35	21 25	20 25	19 35	18 50	17 80	17 15	16 55	20
21	22 40	21 30	20 30	19 40	18 60	17 85	17 20	16 60	21
22	22 45	21 35	20 35	19 50	18 70	17 95	17 30	16 70	22
23	22 55	21 45	20 45	19 55	18 80	18 05	17 40	16 85	23
24	22 65	21 55	20 55	19 70	18 90	18 20	17 55	16 95	24
25	22 75	21 65	20 70	19 80	19 05	18 35	17 70	17 15	25
26	22 85	21 80	20 85	19 95	19 20	18 50	17 90	17 30	26
27	23 00	21 95	21 00	20 15	19 35	18 70	18 05	17 55	27
28	23 20	22 10	21 20	20 35	19 60	18 90	18 30	17 75	28
29	23 35	22 30	21 40	20 55	19 80	19 15	18 55	18 05	29
30	23 60	22 55	21 65	20 80	20 05	19 40	18 85	18 30	30
31	23 85	22 80	21 90	21 10	20 35	19 75	19 15	18 65	31
32	24 10	23 10	22 20	21 40	20 70	20 05	19 50	19 00	32
33	24 40	23 40	22 55	21 75	21 05	20 45	[illegible] 90	19 45	33
34	24 75	23 80	22 90	22 15	21 45	20 85	20 35	19 90	34
35	25 15	24 20	23 35	22 60	21 90	21 35	20 85	20 40	35
36	25 60	24 65	23 80	23 05	22 45	21 85	21 35	20 95	36
37	26 05	25 15	24 35	23 60	23 00	22 45	21 95	21 55	37
38	26 60	25 70	24 90	24 20	23 60	23 10	22 65	—	38
39	27 25	26 35	25 55	24 90	24 30	23 80	—		39
40	27 90	27 05	26 30	25 65	25 10	—			40
41	28 65	27 80	27 10	26 45	—				41
42	29 45	28 65	27 95	—					42
43	30 35	29 55	—						43
44	31 30	—							44

Age nearer Birthday Yrs.	ENDOWMENT TERM 34 Years Rs. P.	35 Years Rs. P.	36 Years Rs. P.	37 Years Rs. P.	38 Years Rs. P.	39 Years Rs. P.	40 Years Rs. P.	41 Years Rs. P.	Age nearer Birthday Yrs.
15	15 75	15 25	14 75	14 35	13 95	13 60	13 30	13 00	15
16	15 75	15 25	14 80	14 40	14 00	13 65	13 35	13 05	16
17	15 80	15 30	14 85	14 45	14 05	13 70	13 40	13 15	17
18	15 85	15 35	14 90	14 50	14 10	13 80	13 50	13 20	18
19	15 90	15 40	14 95	14 55	14 20	13 90	13 60	13 30	19
20	16 00	15 50	15 05	14 65	14 30	14 00	13 70	13 45	20
21	16 10	15 60	15 15	14 75	14 40	14 10	13 80	13 55	21
22	16 20	15 70	15 30	14 90	14 55	14 25	13 95	13 70	22
23	16 30	15 85	15 40	15 05	14 70	14 40	14 15	13 90	23
24	16 45	16 00	15 60	15 20	14 90	14 60	14 30	14 10	24
25	16 65	16 15	15 75	15 40	15 10	14 80	14 55	14 30	25
26	16 80	16 35	16 00	15 60	15 30	15 05	14 80	14 55	26
27	17 05	16 60	16 20	15 85	15 55	15 30	15 05	14 85	27
28	17 30	16 85	16 50	16 15	15 85	15 60	15 35	15 15	28
29	17 55	17 15	16 75	16 45	16 15	15 90	15 70	15 60	29
30	17 85	17 45	17 10	16 80	16 50	16 25	16 05	—	30
31	18 20	17 80	17 45	17 15	16 90	16 65	—		31
32	18 60	18 20	17 90	17 60	17 35	—			32
33	19 00	18 65	18 35	18 05	—				33
34	19 50	19 15	18 80	—					34
35	20 00	19 85	—						35
36	20 55	—							36

The Endowment Insurance Policy (without profits) Table-3 **contd..**

Annual Premiums for the Insurance of Rs. 1000 to be paid at the end of specified term of years or at death, if it should occur sooner

Endowment Term

	42 Years Rs. P.	43 Years Rs. P.	44 Years Rs. P.	45 Years Rs. P.	46 Years Rs. P.	47 Years Rs. P.	48 Years Rs. P.	49 Years Rs. P.	
15	12 75	12 55	12 30	12 15	11 95	11 60	11 65	11 56	15
16	12 80	12 60	12 40	12 20	12 05	11 90	11 75	11 65	16
17	12 90	12 65	12 45	12 30	12 10	12 00	11 85	11 75	17
18	12 95	12 75	12 55	12 40	12 20	12 10	11 95	11 85	18
19	13 05	12 86	12 65	12 60	12 35	12 20	12 10	12 00	19
20	13 20	13 00	12 85	12 65	12 50	12 35	12 25	12 15	20
21	13 35	13 15	12 95	12 80	12 65	12 50	12 40	12 30	21
22	13 50	13 30	13 10	12 95	12 85	12 70	12 60	—	22
23	13 65	13 50	13 30	13 15	13 05	12 90	—		23
24	13 90	13 70	13 55	13 40	13 25				24
25	14 10	13 95	13 80	13 65	—				25
26	14 35	14 20	14 05	—					26
27	14 65	14 50	—						27
28	14 95	—							28

ENDOWMENT TERM

Age nearer Birthday Yrs.	50 Years Rs. P.	51 Years Rs. P.	52 Years Rs. P.	53 Years Rs. P.	54 Years Rs. P.	55 Years Rs. P.	Age nearer Birthday Yrs.
15	11 45	11 35	11 25	11 20	11 10	11 05	15
16	11 55	11 45	11 35	11 30	11 20	—	16
17	11 65	11 55	11 45	11 40	—		17
18	11 75	11 65	11 60	—			18
19	11 90	11 80	—				19
20	12 05	—					20

Concluded

The Endowment Insurance Policy (with profits) Table-4

Annual Premiums for the Insurance of Rs. 1000 to be paid at the end of specified term of years or at death, if it should occur sooner

Age nearer Birthday Yrs.	ENDOWMENT TERM 5 Years Rs P	6 Years Rs P	7 Years Rs. P.	8 Years Rs. P.	9 Years Rs. P.	10 Years Rs. P.	11 Years Rs. P.	Age nearer Birthday Yrs
15	217 15	179 40	152 65	132 70	117 25	107 10	96 65	15
16	217 15	179 40	152 65	132 70	117 25	107 10	96 65	16
17	217 15	179 40	152 65	132 70	117 25	107 10	96 65	17
18	217 15	179 40	152 65	132 70	117 25	107 10	96 65	18
19	217 15	179 40	152 65	132 70	117 25	107 10	96 65	19
20	217 15	179 40	152 65	132 70	117 25	107 10	96 65	20
21	217 15	179 40	152 65	132 70	117 25	107 15	96 65	21
22	217 15	179 40	152 65	132 70	117 25	107 15	96 70	22
23	217 15	179 40	152 65	132 70	117 25	107 15	96 70	23
24	217 15	179 40	152 65	132 75	117 25	107 15	96 70	24
25	217 20	179 40	152 70	132 75	117 30	107 20	96 75	25
26	217 20	179 45	152 70	132 80	117 35	107 25	96 80	26
27	217 20	179 45	152 75	132 80	117 35	107 30	96 85	27
28	217 25	179 50	152 80	132 85	117 45	107 35	96 90	28
29	217 30	179 55	152 85	132 95	117 50	107 40	97 00	29
30	217 35	179 60	152 90	133 00	117 55	107 50	97 10	30
31	217 40	179 70	153 00	133 10	117 65	107 60	97 20	31
32	217 50	179 75	153 10	133 20	117 75	107 75	97 35	32
33	217 60	179 85	153 20	133 30	117 90	107 85	97 50	33
34	217 70	179 95	153 30	133 40	118 05	108 00	97 65	34
35	217 80	180 10	153 45	133 55	118 20	108 20	97 85	35
36	217 90	180 25	153 60	133 75	118 35	108 40	98 05	36
37	218 05	180 40	153 75	133 90	118 55	108 60	98 25	37
38	218 25	180 55	153 95	134 15	118 80	108 85	98 55	38
39	218 40	180 75	154 15	134 35	119 05	109 15	98 80	39
40	218 60	181 00	154 40	134 60	119 30	109 45	99 15	40
41	218 85	181 25	154 65	134 90	119 60	109 75	99 50	41
42	219 10	181 50	154 95	135 20	119 95	110 15	99 90	42
43	219 35	181 80	155 30	135 55	120 30	110 55	100 30	43
44	219 65	182 15	155 65	135 95	120 70	111 00	100 80	44
45	220 00	182 50	156 05	136 35	121 15	111 45	101 30	45
46	220 40	182 90	156 45	136 80	121 65	112 00	101 85	46
47	220 80	183 35	156 95	137 30	122 15	112 60	102 50	47
48	221 25	183 80	157 45	137 85	122 75	113 25	103 20	48
49	221 70	184 35	158 00	138 45	123 40	113 95	103 95	49
50	222 25	184 95	158 65	139 10	124 10	114 75	104 75	50

Age nearer Birthday Yrs.	ENDOWMENT TERM 5 Years Rs. P	6 Years Rs. P.	7 Years Rs. P.	8 Years Rs. P	9 Years Rs. P.	10 Years Rs. P.	11 Years Rs. P.	Age nearer Birthday Yrs.
51	222 85	185 55	159 30	139 85	124 90	115 60	105 65	51
52	223 45	186 25	160 05	140 65	125 75	116 50	106 65	52
53	224 15	187 00	160 85	141 50	126 65	117 55	107 70	53
54	224 95	187 85	161 75	142 45	127 65	118 65	108 90	54
55	225 80	188 75	162 75	143 50	128 75	119 90	110 20	55
56	226 70	189 75	163 80	144 65	130 00	121 20	111 60	56
57	227 70	190 80	164 95	145 90	131 30	122 65	113 10	57
58	228 80	192 00	166 25	147 25	132 75	124 25	114 80	58
59	229 95	193 30	167 60	148 70	134 30	126 00	116 60	59
60	231 25	194 70	169 10	150 30	136 00	127 90	-	60

(Contd.)

The Endowment Insurance Policy (with profits) **Table-4 contd..**

Annual Premiums for the Insurance of Rs. 1000 to be paid at the end of specified term of years or at death, if it should occur sooner

Age nearer Birthday Yrs.	ENDOWMENT TERM							Age nearer Birthday Yrs.
	12 Years Rs. P.	13 Years Rs. P.	14 Years Rs. P.	15 Years Rs. P	16 Years Rs. P.	17 Years Rs. P.	18 Years Rs. P.	
15	87 95	80 60	74 30	70 00	65 10	60 75	56 95	15
16	87 95	80 60	74 30	70 00	65 10	60 75	56 95	16
17	87 95	80 60	74 30	70 00	65 10	60 80	56 95	17
18	87 95	80 60	74 30	70 00	65 10	60 80	56 95	18
19	87 95	80 60	74 30	70 00	65 10	60 80	57 00	19
20	87 95	80 60	74 35	70 00	65 10	60 80	57 00	20
21	87 95	80 65	74 35	70 00	65 15	60 85	57 05	21
22	88 00	80 65	74 35	70 05	65 15	60 85	57 05	22
23	88 00	80 65	74 40	70 10	65 20	60 90	57 10	23
24	88 05	80 70	74 45	70 10	65 25	60 95	57 15	24
25	88 05	80 75	74 50	70 15	65 30	61 05	57 25	25
26	88 10	80 80	74 55	70 25	65 40	61 10	57 35	26
27	88 20	80 85	74 60	70 30	65 45	61 20	57 45	27
28	88 25	80 95	74 70	70 40	65 60	61 30	57 55	28
29	88 35	81 05	74 80	70 65	65 70	61 45	57 70	29
30	88 45	81 15	74 90	70 85	65 85	61 60	57 85	30
31	88 55	81 30	75 05	70 80	66 00	61 75	58 05	31
32	88 70	81 45	75 20	71 00	66 20	61 95	58 25	32
33	88 85	81 60	75 40	71 20	66 40	62 20	58 45	33
34	89 05	81 80	75 60	71 40	66 60	62 45	58 75	34
35	89 25	82 00	75 80	71 65	66 90	62 70	59 00	35
36	89 45	82 25	76 05	71 90	67 15	63 00	59 35	36
37	89 70	82 50	76 35	72 20	67 50	63 35	59 70	37
38	90 00	82 80	76 65	72 55	67 85	63 70	60 10	38
39	90 30	83 10	77 00	72 90	68 25	64 15	60 55	39
40	90 60	83 45	77 35	73 35	68 65	64 60	61 00	40
41	91 00	83 85	77 80	73 80	69 15	65 10	61 55	41
42	91 40	84 30	78 25	74 30	69 65	65 65	62 10	42
43	91 85	84 75	78 75	74 85	70 25	66 25	62 75	43
44	92 35	85 30	79 30	75 45	70 85	66 90	63 45	44
45	92 90	85 85	79 90	76 10	71 55	67 65	64 20	45
46	93 50	86 50	80 60	76 80	72 35	68 45	65 05	46
47	94 15	87 20	81 30	77 60	73 15	69 30	65 95	47
48	94 90	87 95	82 10	78 50	74 05	70 25	67 00	48
49	95 70	88 80	83 00	79 45	75 05	71 30	68 10	49
50	96 55	89 70	83 95	80 45	76 15	72 45	69 30	50

Age nearer Birthday Yrs.	ENDOWMENT TERM							Age nearer Birthday Yrs
	12 Years Rs. P	13 Years Rs. P.	14 Years Rs. P.	15 Years Rs. P.	16 Years Rs. P	17 Years Rs. P.	18 Years Rs. P.	
51	97 60	90 70	85 00	81 60	77 35	73 70	70 60	51
52	98 55	91 80	86 15	82 85	78 65	75 10	72 00	52
53	99 65	93 00	87 40	84 20	80 10	76 60	–	53
54	100 90	94 30	88 80	85 70	81 65	–		54
55	102 25	95 75	90 30	87 35	–			55
56	103 75	97 30	91 95	–				56
57	105 35	99 00	–					57
58	107 15	–	–					58

(Contd.)

Notes

The Endowment Insurance Policy (with profits) Table-4 **contd..**

Annual Premiums for the Insurance of Rs. 1000 to be paid at the end of specified term of years or at death, if it should occur sooner

Age nearer Birthday Yrs.	ENDOWMENT TERM 19 Years Rs. P.	20 Years Rs. P.	21 Years Rs. P.	22 Years Rs. P.	23 Years Rs. P.	24 Years Rs. P.	25 Years Rs. P.	Age nearer Birthday Yrs
15	53 50	50 45	47 85	45 45	43 30	41 30	39 45	15
16	53 50	50 45	47 85	45 50	43 30	41 30	39 50	16
17	53 55	50 45	47 85	45 50	43 35	41 35	39 50	17
18	53 55	50 45	47 90	45 50	43 35	41 40	39 55	18
19	53 55	50 50	47 90	45 55	43 40	41 40	39 60	19
20	53 60	50 50	47 95	45 60	43 45	41 45	39 65	20
21	53 60	50 55	48 00	45 65	43 50	41 50	39 70	21
22	53 65	50 60	48 05	45 70	43 55	41 60	39 80	22
23	53 70	50 65	48 10	45 75	43 60	41 65	39 90	23
24	53 80	50 75	48 15	45 85	43 70	41 75	40 00	24
25	53 85	50 80	48 25	45 95	43 80	41 90	40 10	25
26	53 95	50 90	48 35	46 05	43 95	42 00	40 25	26
27	54 05	51 05	48 50	46 20	44 10	42 15	40 40	27
28	54 20	51 20	48 65	46 35	44 25	42 35	40 60	28
29	54 35	51 35	48 80	46 55	44 45	42 55	40 80	29
30	54 50	51 50	49 00	46 75	44 65	42 80	41 05	30
31	54 70	51 75	49 25	46 95	44 90	43 05	41 35	31
32	54 90	51 95	49 45	47 20	45 20	43 30	41 65	32
33	55 15	52 20	49 75	47 50	45 50	43 65	41 95	33
34	55 45	52 50	50 05	47 85	45 80	44 00	42 35	34
35	55 75	52 85	50 40	48 20	46 20	44 40	42 75	35
36	56 10	53 20	50 75	48 60	46 60	44 80	43 20	36
37	56 45	53 60	51 20	49 00	47 05	45 30	43 70	37
38	56 90	54 05	51 65	49 50	47 55	45 85	44 25	38
39	57 35	54 50	52 15	50 05	48 10	46 40	44 85	39
40	57 85	55 05	52 70	50 60	48 75	47 05	45 55	40
41	58 40	55 60	53 30	51 25	49 40	47 75	46 25	41
42	59 00	56 25	54 00	51 95	50 15	48 50	47 05	42
43	59 70	56 95	54 70	52 70	50 95	49 35	47 90	43
44	60 40	57 75	55 50	53 55	51 80	50 25	48 85	44
45	61 20	58 60	56 40	54 45	52 75	51 25	49 90	45
46	62 10	59 50	57 35	55 45	53 80	52 30	—	46
47	63 05	60 50	58 40	56 55	54 95	—		47
48	64 10	61 60	59 55	57 75	—			48
49	65 25	62 80	60 80	—				49
50	66 50	64 10	—					50
51	67 90	—						51
52	—							52

Age nearer Birthday Yrs.	ENDOWMENT TERM 26 Years Rs. P.	27 Years Rs. P.	28 Years Rs. P.	29 Years Rs. P.	30 Years Rs. P.	31 Years Rs. P.	32 Years Rs. P.	33 Years Rs. P.	Age nearer Birthday Yrs.
15	37 80	36 20	34 75	33 40	32 15	31 10	30 10	29 15	15
16	37 80	36 25	34 80	33 45	32 20	31 15	30 15	29 20	16
17	37 85	36 25	34 85	33 50	32 25	31 20	30 20	29 25	17
18	37 85	36 30	34 90	33 55	32 30	31 25	30 25	29 35	18
19	37 90	36 35	34 95	33 60	32 35	31 30	30 35	29 45	19
20	38 00	36 45	35 00	33 70	32 45	31 40	30 45	29 55	20
21	38 05	36 50	35 10	33 80	32 55	31 50	30 55	29 65	21
22	38 15	36 60	35 20	33 90	32 65	31 65	30 70	29 80	22
23	38 20	36 70	35 30	34 00	32 80	31 75	30 85	29 95	23
24	38 35	36 80	35 45	34 15	32 95	31 95	31 00	30 10	24
25	38 45	36 95	35 60	34 30	33 10	32 10	31 20	30 35	25
26	38 60	37 15	35 75	34 50	33 30	32 35	31 40	30 55	26
27	38 80	37 30	35 95	34 70	33 55	32 55	31 65	30 80	27
28	39 00	37 55	36 20	34 95	33 80	32 85	31 95	31 10	28
29	39 25	37 75	36 45	35 20	34 05	33 10	32 25	31 45	29
30	39 50	38 05	36 70	35 50	34 40	33 45	32 60	31 80	30
31	39 75	38 35	37 05	35 85	34 75	33 80	32 95	32 20	31
32	40 10	38 65	37 40	36 20	35 15	34 20	33 40	32 65	32
33	40 45	39 05	37 75	36 60	35 55	34 65	33 85	33 15	33
34	40 85	39 45	38 20	37 05	36 05	35 15	34 40	33 65	34
35	41 25	39 90	38 70	37 55	36 55	35 70	34 95	34 25	35
36	41 75	40 40	39 20	38 10	37 10	36 30	35 55	34 90	36
37	42 25	40 95	39 80	38 70	37 75	36 95	36 25	35 60	37
38	42 85	41 55	40 40	39 35	38 45	37 65	36 95	—	38
39	43 45	42 20	41 10	40 10	39 20	38 45	—		39
40	44 15	42 95	41 85	40 85	40 00	—			40
41	44 95	43 75	42 85	41 70	—				41
42	45 75	44 60	43 55	—					42
43	46 65	45 55	—						43
44	47 65	—							44
45	—								45

(Contd.)

Notes

The Endowment Insurance Policy (with profits) Table-4 contd..

Annual Premiums for the Insurance of Rs. 1000 to be paid at the end of specified term of years or at death, if it should occur sooner

Age nearer Birthday Yrs.	34 Years Rs. P.	35 Years Rs. P.	36 Years Rs. P.	37 Years Rs. P.	38 Years Rs. P.	39 Years Rs. P.	40 Years Rs. P.	41 Years Rs. P.	Age nearer Birthday Yrs.
15	28 25	27 45	25 65	25 90	25 25	24 60	24 00	23 40	15
16	28 30	27 50	26 75	26 00	25 30	24 70	24 10	23 55	16
17	28 40	27 65	26 80	26 10	25 40	24 80	24 20	23 65	17
18	28 45	27 65	26 90	26 20	25 55	24 90	24 35	23 80	18
19	28 55	27 75	27 00	26 30	25 65	25 05	24 45	23 95	19
20	28 70	27 90	27 15	26 45	25 80	25 20	24 65	24 10	20
21	28 80	28 00	27 30	26 60	25 95	25 35	24 80	24 30	21
22	28 95	28 20	27 45	26 80	26 15	25 55	25 05	24 55	22
23	29 10	28 35	27 65	27 00	26 35	25 80	25 25	24 80	23
24	29 30	28 55	27 85	27 20	26 60	26 05	25 55	25 05	24
25	29 55	28 80	28 10	27 45	26 85	26 30	25 80	25 35	25
26	29 75	29 05	28 35	27 75	27 15	26 65	26 15	25 70	26
27	30 05	29 35	28 65	28 05	27 50	27 00	26 50	26 10	27
28	30 35	29 65	29 00	28 40	27 85	27 35	26 90	26 50	28
29	30 70	30 00	29 40	28 80	28 25	27 80	27 35	26 95	29
30	31 05	30 40	29 80	29 25	28 70	28 25	27 85	—	30
31	31 60	30 85	30 25	29 70	29 20	28 75	—		31
32	31 95	31 30	30 75	30 20	29 75	—			32
33	32 45	31 85	31 30	30 80	—				33
34	33 00	32 40	31 90	—					34
35	33 60	33 05	—						35
36	34 30	—							36

	42 Years Rs. P.	43 Years Rs. P.	44 Years Rs. P.	45 Years Rs. P.	46 Years Rs. P.	47 Years Rs. P.	48 Years Rs. P.	49 Years Rs. P.	
15	22 90	22 40	21 90	21 50	21 10	20 70	20 35	20 00	15
16	23 00	22 50	22 05	21 65	21 25	20 85	20 50	20 20	16
17	23 15	22 65	22 20	21 80	21 40	21 05	20 70	20 40	17
18	23 30	22 80	22 35	21 95	21 60	21 25	20 90	20 60	18
19	23 45	23 00	22 55	22 15	21 80	21 45	21 15	20 85	19
20	23 65	23 15	22 75	22 35	22 00	21 70	21 40	21 10	20
21	23 85	23 40	23 00	22 60	22 25	21 95	21 65	21 40	21
22	24 05	23 65	23 25	22 90	22 55	22 25	22 00	—	22
23	24 35	23 90	23 55	23 20	22 85	22 60	—		23
24	24 60	24 20	23 85	23 50	23 20	—			24
25	24 95	24 55	24 20	23 90	—				25
26	25 30	24 95	24 60	—					26
27	25 70	25 35	—						27
28	26 10	—							28

Age nearer Birthday Yrs.	50 Years Rs. P.	51 Years Rs. P.	52 Years Rs. P.	53 Years Rs. P.	54 Years Rs. P.	55 Years Rs. P.	Age nearer Birthday Yrs.
15	19 70	19 45	19 20	18 95	18 75	18 55	15
16	19 90	19 65	19 40	19 15	18 95	—	16
17	20 10	19 85	19 60	19 40	—		17
18	20 35	20 10	19 85	—			18
19	20 60	20 35	—				19
20	20 85	—					20

Concluded

Notes

Annexure-B

SURRENDER VALUE FACTORS (For Rs. 100 Paid - up)

No. of Yr. Prem. Paid	Term								
	5	6	7	8	9	10	11	12	13
2	61.80	58.13	54.66	51.96	49.56	47.43	45.52	43.83	41.52
3	67.34	63.34	59.55	55.97	53.18	50.70	48.49	46.53	44.78
4	73.39	69.03	64.89	60.98	57.28	54.40	51.84	49.57	47.54
5	-	75.23	70.71	66.44	62.40	58.59	55.62	52.98	50.64
6	-	-	77.06	72.40	67.98	63.82	59.90	56.84	54.12
7	-	-	-	78.90	74.08	69.53	65.24	61.21	58.06
8	-	-	-	-	80.73	75.76	71.08	66.67	62.52
9	-	-	-	-	-	82.57	77.45	72.63	68.09
10	-	-	-	-	-	-	84.40	79.13	74.18
11	-	-	-	-	-	-	-	86.24	80.82
12	-	-	-	-	-	-	-	-	88.07
	14	15	16	17	18	19	20	21	22
2	39.34	37.27	34.62	32.17	29.92	27.84	25.93	24.17	22.55
3	42.42	40.18	37.31	34.66	32.23	29.98	27.91	26.01	24.26
4	45.74	43.32	40.22	37.36	34.72	32.29	30.06	28.00	26.11
5	48.56	46.71	43.35	40.26	37.41	34.78	32.37	30.14	28.10
6	51.72	49.58	46.74	43.40	40.31	37.47	34.86	32.45	30.24
7	55.27	52.79	49.61	46.78	43.45	40.37	37.54	34.94	32.55
8	59.29	56.42	52.82	49.64	46.83	43.50	40.44	37.62	35.03
9	63.83	60.51	56.44	52.86	49.69	46.88	43.57	40.52	37.72
10	69.52	65.15	60.53	56.47	52.90	49.74	46.94	43.64	40.61
11	75.73	70.95	65.17	60.56	56.51	52.94	49.80	47.02	43.74
12	82.50	77.28	70.96	65.19	60.59	56.55	53.00	49.87	47.10
13	89.91	84.19	77.29	70.98	65.22	60.63	56.60	53.07	49.95
14	-	91.74	84.19	77.30	71.00	65.24	60.67	56.66	53.14
15	-	-	91.74	84.20	77.31	71.02	65.28	60.72	56.72
16	-	-	-	91.74	84.20	77.32	71.04	65.31	60.77
17	-	-	-	-	91.74	84.21	77.33	71.06	65.35
18	-	-	-	-	-	91.74	84.22	77.34	71.09
19	-	-	-	-	-	-	91.74	84.22	77.36
20	-	-	-	-	-	-	-	91.74	84.22
21	-	-	-	-	-	-	-	-	91.74
	23	24	25	26	27	28	29	30	
3	22.66	21.18	19.83	18.59	17.46	16.42	15.48	14.62	
4	24.37	22.78	21.31	19.97	18.75	17.63	16.61	15.68	
5	26.22	24.49	22.91	21.46	20.14	18.93	17.82	16.82	
6	28.21	26.34	24.63	23.06	21.63	20.32	19.13	18.04	
7	30.35	28.33	26.48	24.78	23.23	21.82	20.53	19.35	
8	32.65	30.47	28.47	26.64	24.96	23.43	22.03	20.76	
9	35.14	32.78	30.61	28.63	26.82	25.16	23.65	22.27	
10	37.83	35.27	32.92	30.78	28.81	27.02	25.38	23.89	
11	40.72	37.95	35.42	33.09	30.97	29.02	27.25	25.64	
12	43.84	40.84	38.10	35.58	33.28	31.18	29.26	27.51	
13	47.20	43.96	40.98	38.26	35.77	33.49	31.41	29.53	
14	50.04	47.31	44.09	41.14	38.44	35.97	33.72	31.67	
15	53.22	50.14	47.43	44.23	41.31	38.63	36.20	33.98	
16	56.79	53.31	50.25	47.57	44.39	41.60	38.85	36.44	
17	60.82	56.87	53.40	50.37	47.17	44.57	41.70	39.09	
18	65.39	60.89	56.95	53.51	50.50	47.87	44.76	41.92	
19	71.12	65.44	60.95	57.04	53.63	50.65	48.05	44.97	
20	77.38	71.15	65.49	61.03	57.14	53.76	50.81	48.24	
21	84.23	77.39	71.19	65.55	61.11	57.25	53.90	50.98	
22	91.74	84.23	77.41	71.22	65.61	61.20	57.37	54.05	
23	-	91.74	84.24	77.44	71.27	65.68	61.30	57.50	
24	-	-	91.74	84.25	77.46	77.31	65.76	61.41	
25	-	-	-	91.74	84.26	77.49	71.36	65.84	
26	-	-	-	-	91.74	84.27	77.52	71.42	
27	-	-	-	-	-	91.74	84.28	77.55	
28	-	-	-	-	-	-	91.74	84.29	
29	-	-	-	-	-	-	-	91.74	

Annexure-C

REVISED MEDICLAIM PREMIUM SCHEDULE (with 6% loading)

SUM INSURED (overall Liability) (Rs.)	AMOUNT OF LIABILITY FOR DOMICILIARY HOSPITALI-SATION	UPTO 35 YRS	36-45 YRS	46-55 YRS	56-65 YRS	66-70 YRS	71-75 YRS	76-80 YRS
15000	3000	213	232	331	379	427	455	551
20000	4000	275	298	427	490	545	586	730
25000	5000	341	371	537	606	683	730	909
30000	6000	366	398	572	648	724	779	972
35000	7000	390	424	606	689	772	827	1020
40000	8000	543	589	848	965	1075	1151	1419
45000	9000	610	663	951	1082	1213	1295	1585
50000	10000	676	736	1055	1199	1344	1441	1778
55000	11000	749	815	1165	1323	1482	1585	1957
60000	12000	787	854	1220	1392	1557	1667	2053
65000	13000	853	928	1323	1502	1689	1805	2219
70000	14000	920	1001	1427	1620	1813	1943	2412
75000	15000	982	1066	1530	1736	1943	2081	2563
80000	16000	1048	1140	1633	1854	2074	2226	2756
85000	17000	1115	1213	1736	1971	2212	2370	2935
90000	18000	1183	1286	1840	2088	2336	2508	3114
95000	19000	1250	1358	1943	2205	2467	2646	3280
100000	20000	1310	1425	2039	2322	2598	2784	3445
105000	20750	1372	1490	2136	2433	2722	2921	3632
110000	21500	1432	1557	2232	2543	2846	3059	3817
115000	22250	1494	1623	2329	2653	2970	3197	4004
120000	23000	1554	1690	2425	2763	3094	3335	4189

Contd.

REVISED MEDICLAIM PREMIUM SCHEDULE (with 6% loading)

SUM INSURED (overall Liability) (Rs.)	AMOUNT OF LIABILITY FOR DOMICILIARY HOSPITALISATION	UPTO 35 YRS	36-45 YRS	46-55 YRS	56-65 YRS	66-70 YRS	71-75 YRS	76-80 YRS
125000	23750	1615	1755	2522	2874	3218	3473	4376
130000	24500	1676	1822	2618	2984	3342	3610	4561
135000	25250	1737	1888	2715	3094	3466	3748	4748
140000	26000	1798	1955	2811	3204	3590	3886	4933
145000	26750	1859	2020	2908	3315	3714	4024	5120
150000	27500	1920	2087	3004	3425	3838	4162	5305
155000	28250	1975	2147	3094	3528	3955	4293	5484
160000	29000	2030	2206	3183	3632	4073	4423	5664
165000	29750	2085	2266	3273	3734	4189	4555	5843
170000	30500	2139	2326	3362	3838	4307	4685	6022
175000	31250	2194	2385	3452	3941	4423	4817	6201
180000	32000	2249	2444	3541	4045	4541	4947	6380
185000	32750	2304	2505	3632	4148	4658	5078	6559
190000	33500	2359	2564	3721	4252	4775	5209	6738
195000	34250	2414	2624	3811	4354	4892	5340	6918
200000	35000	2469	2683	3900	4458	5010	5471	7097
205000	35500	2518	2736	3982	4555	5120	5595	7269
210000	36000	2566	2789	4065	4651	5230	5719	7441
215000	36500	2615	2842	4148	4748	5340	5843	7614
220000	37000	2664	2895	4230	4844	5451	5967	7786
225000	37500	2713	2948	4313	4941	5561	6091	7958
230000	38000	2761	3001	4398	5037	5671	6215	8130
235000	38500	2810	3054	4479	5134	5781	6339	8303

Contd...

REVISED MEDICLAIM PREMIUM SCHEDULE (with 6% loading)

SUM INSURED (overall Liability) (Rs.)	AMOUNT OF LIABILITY FOR DOMICILIARY HOSPITALISATION	UPTO 35 YRS	36-45 YRS	46-55 YRS	56-65 YRS	66-70 YRS	71-75 YRS	76-80 YRS
240000	39000	2859	3107	4561	5230	5891	6463	8475
245000	39500	2908	3160	4644	5327	6002	6587	8647
250000	40000	2956	3213	4727	5423	6112	6711	8819
255000	40500	3005	3266	4809	5519	6222	6835	8992
260000	41000	3054	3319	4892	5616	6332	6959	9164
265000	41500	3103	3372	4975	5712	6443	7083	9336
270000	42000	3151	3425	5057	5809	6553	7207	9508
275000	42500	3200	3478	5140	5905	6663	7331	9681
280000	43000	3249	3531	5223	6002	6773	7455	9853
285000	43500	3298	3584	5305	6098	6884	7579	10025
290000	44000	3346	3637	5388	6195	6994	7703	10197
295000	44500	3395	3690	5471	6291	7104	7827	10370
300000	45000	3444	3743	5553	6388	7214	7951	10542
305000	45500	3486	3790	5630	6477	7317	8069	10707
310000	46000	3529	3836	5705	6567	7421	8185	10872
315000	46500	3572	3883	5781	6656	7524	8303	11038
320000	47000	3615	3928	5857	6746	7628	8420	11203
325000	47500	3657	3975	5933	6835	7731	8537	11369
330000	48000	3699	4022	6008	6925	7834	8654	11534
335000	48500	3743	4068	6084	7014	7937	8772	11699
340000	49000	3785	4114	6160	7104	8041	8888	11865
345000	49500	3828	4161	6236	7193	8144	9006	12036
350000	50000	3870	4207	6311	7283	8248	9122	12195
355000	50500	3914	4254	6388	7372	8351	9240	12361

Notes

REVISED MEDICLAIM PREMIUM SCHEDULE (with 6% loading)

SUM INSURED (overall Liability) (Rs.)	AMOUNT OF LIABILITY FOR DOMICILIARY HOSPITALI-SATION	UPTO 35 YRS	36-45 YRS	46-55 YRS	56-65 YRS	66-70 YRS	71-75 YRS	76-80 YRS
360000	51000	3956	4299	6463	7462	8455	9357	12526
365000	51500	3998	4346	6539	7551	8557	9474	12691
370000	52000	4041	4393	6614	7642	8661	9591	12857
375000	52500	4084	4439	6691	7731	8764	9709	13022
380000	53000	4127	4485	6766	7821	8868	9825	13187
385000	53500	4169	4532	6842	7910	8971	9943	13353
390000	54000	4211	4578	6918	8000	9075	10059	13518
395000	54500	4255	4625	6994	8089	9177	10177	13684
400000	55000	4297	4670	7069	8179	9281	10294	13849
405000	55500	4340	4717	7145	8263	9384	10411	14014
410000	56000	4382	4764	7221	8358	9488	10528	14180
415000	56500	4426	4810	7297	8447	9591	10646	14345
420000	57000	4468	4856	7372	8537	9695	10762	14510
425000	57500	4510	4903	7449	8626	9798	10880	14676
430000	58000	4553	4949	7524	8716	9901	10996	14842
435000	58500	4596	4996	7600	8805	10004	11114	15000
440000	59000	4639	5041	7675	8896	10108	11231	15172
445000	59500	4681	5088	7752	8985	10211	11348	15337
450000	60000	4723	5135	7827	9075	10315	11465	15503
455000	60500	4767	5181	7903	9164	10418	11583	15668
460000	61000	4809	5227	7979	9254	10522	11699	15833
465000	61500	4852	5274	8055	9343	10624	11817	15999
470000	62000	4894	5320	8130	9433	10728	11933	16164
475000	62500	4937	5367	8207	9522	10831	12051	16329

Concluded

REVISED MEDICLAIM PREMIUM SCHEDULE (with 6% loading)

SUM INSURED (overall Liability) (Rs.)	AMOUNT OF LIABILITY FOR DOMICILIARY HOSPITALISATION	UPTO 35 YRS.	36-45 YRS	46-55 YRS	56-65 YRS	66-70 YRS	71-75 YRS	76-80 YRS
480000	63000	4980	5412	8282	9612	10935	12168	16495
485000	63500	5022	5459	8358	9701	11038	12285	16660
490000	64000	5065	5506	8433	9791	11142	12402	16825
495000	64500	5108	5552	8510	9880	11244	12520	16991
500000	65000	5151	5598	8585	9970	11348	12636	17156

40

INDIRECT TAXES

Government has to perform many functions in the discharge of its duties like infrastructure development, health, education, defence of the country, removal of poverty, maintenance of law and order, etc. To meet these requirements huge amount of capital is required. The question arises, from where does government get money for fulfilling all these activities and for the development of the nation? The government collects money from public through a wide variety of sources i.e. fees, fines, surcharges and taxes which are defined later in this lesson. The most important of these is taxation. In this lesson we will discuss various types of indirect taxes in details.

OBJECTIVES

After studying this lesson, you will be able to:

- acquaint yourself with the sources of revenues of the government;
- define direct taxes and indirect taxes;
- distinguish between direct taxes and indirect taxes;
- state merits and demerits of direct taxes and indirect taxes;
- enumerate sources of direct taxes and indirect taxes;
- define various types of indirect taxes like, excise duty, customs duty(import and export),production linked tax, and Value Added Tax (VAT); and
- distinguish between sales tax and value added tax.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of percentage and its applications

40.1 SOURCES OF REVENUE

As we know that government has to perform its various functions for the welfare of the society, so it requires revenue. The income of government from all sources is called public income or public revenue. Public revenue includes income from taxes, income from goods and services supplied by public enterprises, revenue from the administrative activities, such as fees, fines, etc., gifts and grants, while public receipts include all the income of the government which it may have during a given period of time i.e. public receipts = public revenue + income from all other sources, such as, a public borrowing from individuals and banks and income from public enterprises. Local bodies like Municipal Corporation, Municipal Committees, Town Panchayat, Cantonment Board, etc can also levy certain taxes like property tax, professional tax, octroi, education cess, etc.

Thus, taxes are contributions made by the citizens of the country towards its development and expenditure, which the government has to incur in its social and economic activities. Taxes are paid by the individuals, corporate houses of trade and industry etc. There are different types of taxes like income tax, wealth tax, gift tax, property tax, sales tax, excise and custom duty etc.

40.1.1 Tax

A tax is legally compulsory payment levied by the government on the persons or companies to meet the expenditure incurred on conferring common benefits upon the people of a country. In other words a tax can also be describe as a compulsory levy where those who are taxed have to pay the sums irrespective of any corresponding return of services or goods by the government.

40.1.2 Fee

Fee is also compulsory payment made by a person who receives in return a particular benefit or services from the government.

40.1.3 Fines

These are compulsory payments without any quid pro que but are different from taxes because fines are imposed to curb certain offences and discipline people and not to get revenue for the State. In this sense, fines are not taxes.

40.1.4 Surcharges

It is an additional charge or an extra fee for a special service. It is also called tax on tax e.g. a 10% surcharge is applicable on income tax for incomes above Rs. 10 lakh. In other words surcharges are often a charge in addition to a charge, or a tax added to the original tax.

Two aspects of tax follow from the definition:

1. A tax is a compulsory payment and no one can refuse to pay it.
2. Proceeds from taxes are used for common benefits or general purposes of the state. It means there is no direct quid pro que involvement in the payment of a tax.

Taxes are mainly classified into direct and indirect taxes:

40.2 DIRECT TAXES

Those taxes whose burden cannot be shifted to others and the person who pays these to the government has to bear it are called direct taxes. In other words direct tax is imposed on an individual or a group of individuals, which affects them directly i.e, which they have to pay to the government directly. The direct tax can be of different types:

40.2.1 Income Tax

The tax imposed on an individual or a group of individuals on their annual incomes is known as income tax. Every individual whose annual income exceeds a certain specified limit is required, under the Income Tax Act, to pay a part of his income in the form of income tax. Its rates are announced in the beginning of each financial year by the central government.

Financial Year: The period from 1st April to 31st march is taken as a financial year i.e. every financial year begins on 1st April and ends on 31st march of the consecutive year.

Assessment Year: The year next to a particular financial year is called the assessment year for that financial year, e.g. for financial year 2005-06, the assessment year is 2006-07.

Permanent Account Number: An individual is given a permanent account number (PAN) by the income tax department. He or she is obliged to file an income tax return of the financial year by a specified date of the subsequent financial year.

40.2.2 Wealth Tax

The tax imposed on the wealth (property as well as money) of an individual is called wealth tax. The exemption limit for wealth tax is Rs 5, 00,000. In addition one residential house or a part thereof is exempted from the wealth tax.

40.2.3 Gift Tax

If an individual transfers any of his movable or immovable property voluntarily to any other individual it is called a gift. If the value of a gift exceeds a specified limit then the person giving the gift has to pay gift tax to the government where as the person receiving the gift need not pay any tax.

A controversial issue in public finance is concerned with whether in the tax structure of an economy, direct or indirect tax should be preferred. Indeed both direct taxes and indirect taxes have their merits and demerits and therefore a good tax system should contain a proper mix of these two types of taxes.

Direct taxes, it may be recalled are those which are levied directly on the individuals and firms and their burden is borne by those on whom these are levied.

40.2.4 Merits of Direct Taxes

1. The larger burden of the direct taxes falls on the rich people who have capacity to bear these and the poor people with less ability to pay have to bear less burden.
2. Direct taxes are important instrument of reducing inequalities of income and wealth.
3. Unlike indirect taxes, direct taxes do not cause distortion in the allocation of resources. As a result these leave the consumers better off as compared to indirect taxes.
4. Revenue elasticity of direct taxes, especially if they are of progressive type is quite high. As the national income increases, the revenue on these taxes also rises a great deal.

40.2.5 Demerits of Direct Taxes

1. In the direct taxation, people are aware of their tax liability and therefore they would try to avoid or even evade the taxes. The practice and possibility of tax evasion and avoidance

is more in direct taxes than in case of indirect taxes.

2. Direct taxes are generally payable in lump sum or even in advance and become quite inconvenient.
3. Another demerit of direct taxes is their supposed effect on the will to work and save. It is assessed that work (given Income) and leisure are two alternatives before any taxpayer. If therefore, a tax is imposed say on income, the taxpayer will find that the return from work has decreased as compared with return from leisure. He therefore tries to substitute leisure for work.

40.3 INDIRECT TAXES

Indirect taxes are those whose burden can be shifted to others so that those who pay these taxes to the government do not bear the whole burden but pass it on wholly or partly to others. Indirect taxes are levied on production and sale of commodities and services and small or a large part of the burden of indirect taxes are passed on to the consumers. Excise duties on the product of commodities, sales tax, service tax, customs duty, tax on rail or bus fare are some examples of indirect taxes.

40.3.1 Excise Duty

The tax imposed by the government on the manufacturer or producer on the production of some items is called excise duty. The liability to pay excise duty is always on the manufacturer or producer of goods. The duty being a duty on manufacture of goods, it is normally added to the cost of goods, and is collected by the manufacturer from the buyer of goods. Therefore it is called an indirect tax. This duty is now termed as "Cenvat". There are three types of parties who can be considered as manufacturers-

- Those who personally manufacture the goods in question
- Those who get the goods manufactured by employing hired labour
- Those who get the goods manufactured by other parties

For example, excise duty on the production of sugar is an indirect tax because the manufacturers of sugar include the excise duty in the price and pass it on to buyers. Ultimately it is the consumers on whom the incidence of excise duty on sugar falls, as they will pay higher price for sugar than before the imposition of the tax.

In order to attract Excise duty liability, following four conditions must be fulfilled:

a) The duty is on "goods".
b) The goods must be "excisable"
c) The goods must be "manufactured" or produced.
d) Such manufacture or production must be "in India".

40.3.2 Additional Information on Excise Duty

Goods : These are the entities, which can be weighted, measured and marketed. e.g. steel, cloth, computer software, gas, etc. Those commodities having very short life are not goods, if not marketable in that short period, even if there is a specific entry in the tariff.

Excise duty can only be levied on those items, which are manufactured in India but excluding goods produced or manufactured in Special Economic Zones (SEZ). Thus, excise levy cannot be imposed on imported goods.

Payment of excise duty : In case of Non-SSI (Small Scale Industries) i.e., normal assesses the excise duty is payable monthly, and for SSI (availing exemption based on turnover) it is payable quarterly. The duty on the goods removed from the factory or the warehouse during the month shall be paid by the 5th of the following month in case of Non-SSI and by 15th for SSI. In case of delayed payment, interest should also be deposited at the rate of 13% p.m or Rs 1,000 per day for the period of delay after 5th or 15th whichever is applicable, whichever is higher, along with the duty.

Payment by debit in Cenvat credit account: Under the Cenvat credit scheme, the assessee is allowed credit of duty paid on inputs or capital goods, which are used in or in relation to manufacture of the final products, and the credit can be utilized towards payment of duty on the final products. Credit is allowed on inputs and capital goods except LDO (light diesel oil), HSD (high speed diesel) and motor spirit. Also, instant credit is allowed immediately on the inputs being received into the factory. However credit is not allowed if final products are exempted from duty.

Following example will illustrate the credit method of Cenvat.

Let the price of the commodity be Rs 100, When the transaction takes place without cenvat, B purchases from A at Rs 110,(10% as excise duty). After addition a value of Rs 40, the subtotal is Rs 150.He pays 10% tax on it (i.e Rs15) then total is 165. As against this, in the second case, when transaction takes place with Cenvat, B purchases from A at Rs 100 because he got credit on that amount. After adding the same value of Rs 40, the sub total is Rs 140, He has to pay 10% of excise on Rs 140,i.e Rs 14, then total becomes Rs. 154. Here you can observe easily that transaction with Cenvat is clearly beneficial. The details are exhibited in the following tabular form:

	Transaction without Cenvat		**Transaction with Cenvat**	
Details	A	B	A	B
Purchases	-	110	-	100
Value added	100	40	100	40
Sub-total	100	150	100	140
Add-tax 10%	10	15	10	14
Total	110	165	110	154

Exemption from Payment of Excise Duty: Central excise rules grant exemption from duty if goods are exported under bond, except exports to Nepal and Bhutan. Similarly, goods manufactured in Special Economic Zones (SEZ) are not excisable and hence no excise duty can be levied on goods manufactured in SEZ. Certain other items, which are exempted for excise duty, are enlisted in **Annexure-'A'**, given at the end of this lesson.

Generally 16% excise duty and 2% cess on it are imposed on most goods, but government can fix different tariff values for different classes of goods or goods manufactured by different classes or sold to different classes of buyers. Few exceptions like the following are there in case of Textile sector.

Notes

Sl. No.	Type of Goods	Excise Duty	Cess
1.	Unprocessed fabrics of cotton, man-made (synthetic) and woolen other than (2) given below.	10%	2%
2.	Unprocessed knitted or crocheted fabric of cotton not containing any other textile materials.	8%	2%
3.	If readymade garments are made up of 100% cotton fabrics and also knitted or crocheted	8%	2%
4.	Readymade garments other than (3) above	10%	2%
5.	All types of clothing accessories if made up of 100% cotton and also knitted or crocheted.	8%	2%
6.	Clothing accessories other than (5) above.	10%	2%

40.3.3 Valuation for Excise Duty

Specific duty: It is the duty payable on the basis of certain unit e.g. duty on cigarettes is on length basis, sugar per quintal basis, matches per 100 boxes, marble slabs and tiles per square meter basis and colour TV by screen size in cm, if MRP is not written on the carton.

Tariff Value: Government from time to time fixes tariff value. Government can fix different tariff values for different classes of goods manufactured by different classes or sold to different classes of buyers.

MRP based valuation : The provisions are as follows:

i) The goods should be covered under provisions of Standards of Weights and Measures Act.

ii) Central Government can permit reasonable abatement (deductions) from the retail sale price.

iii) Central Government has to issue a notification in Official gazette specifying the commodities for which the provision is applicable and the abatement permissible.

For example, government had issued a notification to reduce the excise duty on cosmetics and toilet preparations on MRP basis printed on the carton after allowing abatement of 50%. In such cases, if MRP printed on carton is Rs 50 and if the duty on cosmetics & toilet preparations is 20%, the duty @ 20% will be payable on Rs 25 (i.e after allowing 50% abatement of MRP of Rs 50). Thus duty payable per pack will be Rs. 5.00.

Assessable Value: The basic provision of assessable value, when excise duty is chargeable on excisable goods with reference to value will be transaction value on each removal of goods. Transaction value is defined as the price actually paid or payable for the goods, when sold and includes in addition to the amount charged as price, any amount that the buyer is liable to pay, including any amount charged for advertising or publicity, marketing and selling organization expenses, storage, outward handling, servicing, warranty, commission or any other matter, but does not include the amount of excise duty, sales tax and other taxes.

40.3.4 Export benefits under Central Excise

Inputs free of duty: Exporting units need raw materials without payment of customs/excise duty, to enable them to compete for exporting.Exports free of duty on finished product: exports of almost all excisable goods except hides, skins and leather and salt and exports to all countries except to Nepal and Bhutan are exempted from central excise duties.

Notes

Example 40.1 Shivam Enterprises, manufactures 60 units of steam irons per day, and its input cost is Rs.200 per unit. The company adds a value of Rs.100 and then sells it after paying 10% excise duty. Calculate the final price of each steam iron, and how much total duty has been paid at the end of the month when the transaction is without Cenvat.

Solution : Input Cost = Rs. 200 per unit

Value added = Rs.100

Total = Rs. 300

for 60 units per day in a month = 60 x 30 = 1800 units

Duty paid $= Rs.300 \times 1800 \times \frac{10}{100}$ = Rs. 54,000

Example 40.2 Ganesh and Sons, produce 100 kgs chocolate biscuits per day at the cost of Rs. 50 per kg. If the excise duty is 5%, then how much duty has to be paid at the end of the month, if Rs. 20 per kg is added to the cost.

Solution: Input Cost per kg. = Rs. 50

Value added = Rs. 20

Total = Rs. 70

Duty for one month $= Rs.[70 \times 100 \times 30 \times \frac{5}{100}]$ = Rs. 10500

Example 40.3 Sharma and Company manufactures 5 quilts a day and uses cotton fiber (Rs.100 per kg) and cotton cloth (Rs.50 per meter) as input. In making one quilt 2 kgs of cotton fibre and 5 meters of cloth are used. If excise duty on cotton fiber is 8% and on cloth it is 10% while on quilt it is 12%, calculate the total duty paid to the government in one month when the value added by the Company is Rs.109 per quilt, if the transaction is (i) **without Cenvat** (ii) **with Cenvat**.

Solution :Input cost of 1 quilt $= Rs. (2 \times 100 + 5 \times 50)$

= Rs. 200 on cotton fiber + Rs. 250 on cloth.

∴ (i) Without Cenvat (for one quilt)

Cotton Fibre =	Rs. 200	Cloth =	Rs. 250
	+ Rs. 16		+ Rs. 25
	Rs. 216		Rs. 275

Total for 1 quilt = Rs. (216 + 275) = Rs. 491

Value added = Rs.109
Total = Rs. 600
Duty 12% = Rs. 72
Total = Rs. 672

and excise duty paid for 1 month @ 5 quilts per day

$$= 5\times30\times[16+25+72] = \text{Rs.}16950$$

(ii) With Cenvat (for one quilt)

Input cost = Rs. (200 + 250) = Rs. 450
Value added = Rs. 109
Total = Rs. 559
Duty (12%) = 67.08
Total = 626.08
Total duty paid for one month = $5\times67.08\times30 =$ Rs. 10,062

Example 40.4 Mrs. Ahuja's unit stiches 20 ladies suits out of which 50% she exported to America and the rest 50% are sold in the domestic market. In preparing one suit,5 meters cloth is required which is purchased at Rs.120 per meter and she adds value of Rs.100 per suit. If excise duty on cloth is 5% then calculate how much excise duty she has to pay to the government at the end of the month using Cenvat transaction method.

Solution : Input cost for one suit = $Rs.\ 5\times120 = Rs.\ 600$
Value added = Rs.100
Total = Rs. 700
Duty (5%) = Rs. 35
Total = Rs. 735
Duty paid for 1 month = Rs. $10\times30\times35 = Rs.10500$

[∵ 50% are exported on which, no duty is paid]

Example 40.5 A sugar mill produces 10,000 kgs of sugar per day at an input cost of Rs. 20 per kg and adds a value of Rs.10 per kg. The excise duty is 5% which is to be paid on 5th of the following month. If delayed a penalty of 13% per month or Rs.1000 per day, whichever is higher is to be paid. The Company paid the duty on 20th day of the following month. How much duty has to be paid with Cenvat.

Solution: Input cost per kg = Rs. 20
Value added = Rs.10
Total = Rs. 30
Duty (5%) = Rs. 1.50
Total = Rs. 31.50
Total Duty for 1 month = 10,000 ×1.50 ×30 = Rs. 4,50,000

penalty $= Rs.\ [450,000\times\frac{13}{100}\times\frac{15}{30}] = Rs.\ 29250$

Total paid = Rs. [450000+29250] = Rs. 479250

Example 40.6 Mr. Gowda and Company manufactured 2000 tons of steel in a month at the rate of Rs. 18000 per ton, excise duty is 16% with 2% education cess. The Company exported half of the production to Uganda. The Company could not pay the duty on time, how much duty will have to be paid at the end of 20 days after the due date.

Solution : Quantity produced per month = 2000 tons

Quantity eligible for duty = 1000 tons

Cost = Rs.1000 × 18000 = Rs. 18,00,0000

Duty (16%) = Rs. 2880000

Education cess (2%) = Rs. 57600

Total = Rs. 2937600

Penalty $= Rs.\ 2937600 \times \frac{13}{100} \times \frac{20}{30} = Rs.\ 254592$

Total to be paid = Rs. [2937600 + 254592]

= Rs. 3192192

Example 40.7 Singh and Company produces steel utensils at the rate of 200 kgs per day after using inputs of Rs.10,000. The excise duty is 16% with 2% education cess. If the duty is paid 10 days after the due date, calculate the amount of penalty to be paid.

Solution : Inputs = Rs.10000

Duty (16%) = Rs.1600

Education cess = Rs. 32

Total = Rs.1632

Penalty $= \text{Rs. } 1632 \times \frac{13}{100} \times \frac{10}{30} = Rs.\ 70.72$

Which is less than Rs. 10 × 1000 = Rs.10000 (@Rs.1000 per day)

Penalty to be paid is Rs.10,000

Example 40.8 Singla Enterprises produces 10 kgs. of wafers per day by using inputs of Rs.150. The company added a value of Rs.10 per kg. The excise duty is 16%. Calculate the total excise duty paid after a month through (i) **with Cenvat** (ii) **without Cenvat.**

Solution: (i) With Cenvat

Input Cost = Rs.150

Value added = 10×10 = Rs.100

Total = Rs.250

Duty (16%) = Rs. 40

Total = Rs.290

∴ Total duty paid per month = Rs. 40×30

=Rs.1200

(ii) Without Cenvat

Input Cost = Rs.150
Duty (16%) = Rs. 24
Total = Rs. 174
Value added = Rs.100
Total = Rs. 274

Duty (16%) = Rs. 43.84
Total = Rs. 317.84
Total duty paid = Rs. $(43.84 + 24.00) \times 30$
= Rs. 2035.20

∴ With Cenvat is beneficial to the customer.

CHECK YOUR PROGRESS 40.1

1. A cotton mill, manufactures 100 kgs of cotton and 118 kgs of nylon per day. The cost of production of cotton and nylon is Rs.18.50 per kg, and Rs. 23.69 per kg respectively. If the excise duty on cotton is 8% and on nylon is 10%, how much excise duty, the mill has to pay at the end of each month?
2. Kohli Garments, manufactures readymade garments. It utilizes 50 meters of cloth per day which is Rs. 60 per meter. From 50 meters of cloth, it produces 30 frocks which are sold after adding a value of Rs.50 for each frock. If the excise duty on frocks is 10%, calculate the total excise duty, the company has to pay to the government per month, if the transaction is with Cenvat.
3. Gupta and Sons produces plastic bags in two factories. One is in Special Economic Zone (SEZ). Both factories produce equal amount of plastic bags i.e. 100 kgs per day. If the excise duty levied is 20%, calculate the excise duty paid at the end of the month, when the company is selling these bags at a price of Rs.60 per kg including the excise duty paid.
4. Mrs. Mehta in her factory manufactures biscuits at the rate of 1000 kgs. per day. She fixes Rs.80 per kg. as MRP and gets abatement of 50% on biscuits. How much duty she has to pay at the end of the month if excise duty on biscuits is 16%.
5. A company manufactures 10 refrigerators per day at the rate of Rs. 21500. Duty to be paid is 10% with abatement of 20%. How much duty the company has to pay at the end of the month?

40.3.5 Customs Duty

Custom duty is a form of indirect tax. Standard English dictionary defines the term "custom" as duties imposed on imported or less commonly exported goods. This term is usually applied to those taxes which are payable upon goods or merchandise imported or exported. It is also defined as tax imposed by the government on the import of items (goods). The Customs Act was formulated in 1962 to prevent illegal imports and exports of goods. Besides, all imports are sought to be subject to a duty with a view to affording protection to indigenous industries.

40.3.6 Additional Information on Customs Duty

Education cess @2% : With effect from 10.09.2004 an education cess has been levied on items imported into India. It is leviable @2% on the aggregate of customs duties leviable on such goods.

No duty on pilfered goods: If any imported goods are pilfered after the unloading thereof and before the proper officer has made on order for clearance for home consumption or deposit in a warehouse, the importer shall not be liable to pay the duty leviable on such good. The term "pilfer" means to steal especially in small quantities.

Abatement of duty on damaged goods: The term 'damage' denotes physical damage to the goods. This implies that the goods are not fit to be used for the purpose for which they are meant. The damaged goods get some % of abatement of damage in the customs duty.

Exemption: Article 265 of the Indian Constitution provides that 'no tax shall be levied or collected except by authority of law. The power of the central government to alter the duty rate structure is known as delegated legislation and this power is always subject to superintendence and check by parliament. If the central government is satisfied that it is necessary in the public interest so to do, then whole or part of customs duty can be exempted from the customs duty.

Additional duty of Customs: Apart from the customs duty levied as a percentage of the value of goods, the following example illustrates the method of computing the additional duty of customs.

Assessable value	: Rs 1,000
Rate of basic customs duty	: 25%
Rate of additional customs duty	: 16%
Basic customs duty @25% of Rs1000	: Rs 250
Total value for computing additional customs duty	: Rs 1250
Additional customs duty (16% on Rs1250)	: Rs 200
Total duty payable 250+200	: Rs 450

Example 40.9 Chaddha & Chaddha group of companies imports steel worth Rs 10 crores and customs duty levied on steel is 10%. Calculate the total amount of custom duty they have to pay on this transaction, if 2% education cess is to be charged on customs duty.

Solution : Total cost of imported steel is Rs 10 crore i.e. 100000000

Custom duty imposed @ 10%

$$= 100000000 \times \frac{10}{100}$$

$$= 10000000$$

$$\text{Education cess} = 10000000 \times \frac{2}{100} = Rs.\ 200,000$$

Total duty paid by the company is Rs. 102,00,000.

Example 40.10 Ram Kumar imports 25 quintals of fiber from England at the rate of £500 per quintal. Customs duty levied on fiber is 40%. Calculate how much he has to pay (in rupees) to the government as customs duty, if 2% education cess is to be charged on customs duty. (Take £ 1 = Rs. 80)

Solution : Total cost of fiber = Rs. 25×500×80

= Rs. 10,00,000

customs duty $= Rs. 10,00,000 \times \frac{40}{100}$

= Rs. 400,000

Education cess (2%) = Rs. 8000

Total amount to be paid = Rs. 408000

Example 40.11 Mr. Prasad imports 700 kgs of sugar per day @ Rs 20 per kg, Customs duty on sugar is 20%. How much customs duty he has to pay in a month, if 2% education cess is to be charged on customs duty.

Solution: Total cost per day = Rs.[700 × 20] =Rs.14000

Customs duty $= 14000 \times \frac{20}{100} =$ Rs. 2800

Customs duty for the month = Rs. 2800 × 30

= Rs. 84000

Education cess (2%) = Rs.1680

Total = Rs. 85680

Example 40.12 Mr. Kumar imports 50 kgs of chocolate @ Rs 250 per kg, 80 kgs of biscuits @ Rs 400 per kg and 1 quintal of wafers @ Rs 200 per kg. 25% of chocolate, 10% of biscuits and 15% of wafers were damaged in the transport. Customs duty on all these items is 25% but on damaged goods it is 5%. Calculate the total amount of duty he has to pay for this transaction, if 2% education cess is to be charged on customs duty.

Solution:

Damaged Items	**Undamaged Items**
$\text{Chocolates} = 50 \times \frac{25}{100} = 12.5kg$	Chocolates = 37.5 kg
$\text{Biscuits} = 80 \times \frac{10}{100} = 8kg$	Biscuits = 72kg
$\text{Wafers} = 100 \times \frac{15}{100} = 15kg$	Wafers = 85kg

Cost = $Rs.\ (12.5 \times 250 + 8 \times 400 + 15 \times 200) = Rs.9325$ (damaged items)

Cost = $Rs.\ (37.5 \times 250 + 72 \times 400 + 85 \times 200) = Rs.55175$ (undamaged items)

$\therefore$ Duty $= Rs[9325\times\frac{5}{100}+55175\times\frac{25}{100}] = Rs.14726.25$

Education cess = 295

Total =15021.25

Example 40.13 Mr. Mohta imports 5 T.V. sets@ \$240 per set, 10 dish washers @ \$400 per unit and 25 computers @\$500 per unit from Japan. Customs duty on TV is 25%, on dish washer 30% and computer is exempted from customs duty. Calculate the total amount of customs duty (in rupees) he has to pay, if 2% education cess is to be charged on customs duty. (Take \$1=Rs 45).

Solution: Customs Duty on 5 TV sets

$$= Rs.\ [5\times 240\times 45\times\frac{25}{100}] = Rs.\ 13500$$

customs duty on 10 dish washer

$$= Rs.\ [10\times 400\times 45\times\frac{30}{100}] = Rs.\ 54000$$

Customs duty on computers = NIL

Total duty paid = Rs. 67500

Education cess (2%) = Rs. 1350

Total = Rs. 68850

Example 40.14 Mr. Gupta imports 20 Quintals of packed food @ Rs. 20 per kg. Customs duty imposed on it is 25% and 16% of additional duty. Calculate the total amount Mr Gupta has to pay to the government, if 10% of the goods were pilfered. Assume that 2% education cess is charged on customs duty.

Solution: Quantity pilfered = 2 quintals

Balance = 18 quintals

Cost $= Rs.\ [18\times 2000] = Rs.\ 36000$

Customs duty (25%) = Rs. 9000

Total = Rs. 45000

Additional duty (16%) $= Rs.\ 45000\times\frac{16}{100} = Rs.\ 7200$

Total duty paid = Rs. (9000 + 7200)

= Rs.16200

Education cess (2%) = Rs. 324

Total = Rs.16524

Example 40.15 Mr. Khurana imports 200 kgs of cashew nuts in which 50% are shelled and 50% are inshelled and 100 kgs of almonds out of which 50% are shelled and 50% are inshelled.

Customs duty on shelled nuts is 70% and 60% in case of inshelled nuts. Inshelled cashew nut is Rs 200 per kg and shelled is Rs 250 per kg and inshelled almond is Rs 300 per kg and shelled is Rs 350 per kg. If 25% cess is levied as additional duty on shelled nuts, calculate how much Mr Khurana has to pay as customs duty, if 10% of the goods were pilfered. Assume that 2% education cess is charged on customs duty.

Solution: Cost of shelled cashew nuts = Rs. (100 × 250) = Rs. 25000

Duty $= Rs.\ 25000 \times \frac{70}{100} = Rs.\ 17500$

Total = Rs. 42500

Additional duty (25%) $= Rs.[42500 \times \frac{25}{100}] = Rs.10625$

Total duty paid in this case = Rs. 28125 ....(i)

Duty on inshelled cashew nuts $= Rs.100 \times 200 \times \frac{60}{100} = Rs.12000$ ...(ii)

Price of shelled almonds = Rs. 350 × 50 =Rs.17500

Duty (70%) $= Rs.\ 17500 \times \frac{70}{100} = Rs.\ 12250$

Total = Rs. 29750

Additional duty (25%) $= Rs.\ (29750 \times \frac{25}{100}) = Rs.\ 7437.50$

Total duty on shelled Almonds = Rs.19687.50 ....(iii)

Duty on inshelled almonds $= Rs.\ [300 \times 50 \times \frac{60}{100}] = Rs.9000$ ...(iv)

Adding (i), (ii), (iii) and (iv) we have

Amount = Rs.68812.50

Education Cess = Rs. 1376.25

Total $= Rs.\ 70188.75 \approx Rs.\ 70189$

CHECK YOUR PROGRESS 40.2

1. Kishori Lal & Sons, imports 100 quintals of wheat @ Rs 1000 per Quintal and Customs duty levied on wheat is 35%. If 5% of the import items are pilfered then how much duty they have to pay, if 2% education cess is to be charged on customs duty.
(Note: No customs duty is paid on the pilfered goods).
2. Mr. Gulati imports 90 kgs of wheat @ Rs 10 per kg and 120 kgs of rice @ Rs 20 per kg. The customs duty on import of wheat is 35 % as against 20% for rice. Find the total duty paid, if 2% education cess is to be charged on customs duty.
3. Moti Lal & Company imports capital equipments worth Rs 10,000 crores. He supplies these goods to public as well as to private enterprises on 50:50 basis. For public enterprises customs duty is 30% but it is double in case of private enterprises. Calculate how much

customs duty he has to pay in total, if 2% education cess is to be charged on customs duty.

4. Mr. Mittal imported 2500 quintals of wheat @ Rs. 12 per kg, 20% of customs duty and 2% educational cess is levied on it. If 2% of the wheat was pilfered and 5% are damaged on which 5% duty is levied, calculate the total amount payable to the government?

Notes

40.3.7 Sales Tax

Tax paid by the consumer on the purchase of some items is called the sales tax. Rates of sales tax depend upon the nature of the goods purchased by the consumer.

40.3.8 Value Added Tax

Under the Indian constitution, the States have the exclusive powers to levy tax on the sales of goods. The tax on the inter-state trade is levied by central government, and is called Central Sales Tax (CST). It is proposed to abolish CST in phased manner. Due to various defects in the Sales Tax System, the Govt, has introduced a new system called Value Added Tax (VAT) in place of State Sales Tax.

VAT is a multi-point tax levied and collected on the value added to goods at different stages of sale. It is a method of taxing by stages. The method consists of levying a tax on the value added to a product at each stage of production or distribution. It is another form of sales tax where tax is collected in stages rather than collection of the tax at the first or last point. VAT, in simple terms, is a multi-point levy on each of the entities in the supply chain with the facility of set-off of input tax i.e. that is, the tax paid at the stage of purchase of goods by a trader and on purchase of raw materials by a manufacturer. Only the value addition in the hands of each of the entities is subject to tax. For instance, if a dealer purchases goods for Rs 100 from another dealer and a tax of Rs 10 has been charged in the bill, and he sells the goods for Rs 120 on which the dealer will charge a tax of Rs 12 at 10 per cent, the tax payable by the dealer will be only Rs 2, being the difference between Rs. 12 the tax collected and Rs. 10 tax already paid on purchases. Thus, the dealer has paid tax at 10 per cent on Rs 20 being the value addition of goods in his hands.

Most State governments have implemented VAT w.e.f. 1.4.2005. Haryana was the first state to implement VAT w.e.f. 1.4.2004 in the first year itself, Growth in tax revenue has been reported by the States as compared to the tax collection during the same period in previous year. In case of the loss to the State on switching over to VAT, the Central Government will compensate the loss to the State.

40.4 CHARACTERISTICS OF VAT

1. It is simple, modern and transparent tax system.
2. It is a multipoint tax with credit for the tax paid at preceding stage.
3. Small traders (whose turnover is up to Rs10 lakhs) are outside VAT.
4. VAT replaces a number of taxes like turnover tax, luxury tax, surcharge etc.
5. VAT being efficient is considered to be better than sales tax.
6. VAT has four rates instead of the large number of rates under sales tax.
7. Composition scheme for small dealer having turnover above taxable quantum of Rs 10 lakhs but below 50 lakhs.

8. VAT eliminates cascading by providing credit of taxes paid on inputs and only taxing value addition.

40.4.1 Calculation of Tax Liability under VAT

Suppose a TV dealer sells TV worth Rs. 20,000 and VAT is 4%, he will collect Rs. 800 (20,000×0.04) as VAT. If the dealer had purchased the TV for Rs. 19,000 and at that time he had already paid Rs. 760 as VAT. So the VAT payable by the dealer will be 800 – 760 = 40. He will pay to the government only Rs. 40.00 the tax payable is tax rate multiplied by valuation addition. In this case it would be $0.04 \times (20000 - 19000) = 40$.

VAT liability for any tax period, is calculated by decreasing total input tax from total output tax. The output tax is calculated by multiplying the turn over (Sales) by applicable VAT rates.

$$\textit{Net tax} = \textit{output tax} - \textit{input tax}$$

If difference is (+) pay this amount to government.

If difference is (–) apply excess credit against your VAT liability and claim refund for any remaining balance OR the excess credit can be carried forward to the next period.

40.4.2 Advantages of VAT

1. Self-assessment by dealers.
2. Higher revenue growth from states.
3. Set off for input tax paid on previous purchases.
4. Other taxes to be eliminated.
5. Fairness in the taxation system. Visits to tax department will reduce.
6. Help to reduce tax evasion and corruption.
7. Uniform rates of VAT will boost fair trade.
8. VAT does not lead to price rise.
9. VAT is easier to enforce.

40.4.3 Disadvantages of VAT

1. Record keeping systems and procedure will need to re-strengthen with Tax Authorities in order to claim input tax credit.
2. VAT may lead to tax evasion if false input credits are submitted by dealers.

40.4.4 Additional Information on VAT

Turnover: It means the aggregate of the amounts of purchase price paid or payable by a person in any tax period, including any input tax.

Sale: Any transfer of property in goods by one person to another for cash or for any deferred payment.

Rates of Tax: The rates of VAT payable on the taxable turnover of a dealer shall be :

i) in respect of goods specified in the second schedule, at the rate of 1%.

ii) in respect of goods specified in the third schedule, at the rate of 4%.

iii) in respect of goods specified in the fourth schedule, at the rate of 20%.

iv) and all the goods other than those in three schedules, at the rate of 12.5%.

Tax Credit: A dealer who is registered shall be entitled to a tax credit in respect of the turnover of purchases occurring during the tax period where the purchase arises in the course of his activities as a dealer and the goods are to be used by him directly or indirectly for the purpose of making sale.

No tax credit shall be allowed

i) in the case of the purchase of goods from a person who is not a registered dealer.
ii) for the purchase of goods which are to be incorporated into the structure of a building owned or occupied by the person.
iii) when a dealer has purchased goods and the goods are to be used partly for the purpose of making the sales, the amount of the tax credit shall be reduced proportionately.

Net Tax: The net tax payable by a dealer for a tax period shall be determined by the formula:

$$\textbf{Net tax} = O - I - C$$

where

O= the amount of tax payable by the person at rates stipulated in respect of the taxable turnover arising in the tax period.

I= the amount of the tax credit arising in the tax period to which the person is entitled for adjustment to the tax credit required by this Act.

C= the amount if any, brought forward from the previous tax period.

Penalty: If a person is required to furnish a return, but fails to furnish any return by the due date or fails to furnish with a return any other document that is required to be furnished with the return then he has to pay penalty of Rs 100. per day from the day on which the requirement arose until the failure is rectified and maximum amount of this penalty is Rs. 10,000.

Example 40.16 If 'A' purchases goods worth Rs. 20,000 from the manufacturer and adds value of Rs. 5,000, calculate the total sale price of the product, if VAT levied @ 12.5%.

Solution: Cost price = Rs. 20,000

Value added = Rs. 5000

VAT (12.5%) = $Rs.\ 5000 \times \frac{125}{1000} = Rs.\ 625$

Total sale price = Rs. 25625

Example 40.17 Ms. Raghava purchases cotton fiber @ Rs. 50 per kg and 1 kg of fiber produces 2 meters of cloth. She again sold this cloth in the market @ Rs. 38.50 per meters, VAT levied on the cloth is 8%. Calculate the total VAT collected by the govt. in this whole transaction?

Solution: Cost of cotton fiber = Rs. 50

Selling price of cloth = Rs.77

Difference = Rs.27

$\therefore$ 27 = value added + VAT

$$= x + \frac{8}{100}.x = \frac{27x}{25}$$

or, $x = 25$ i.e value added = Rs. 25

VAT= Rs. 2

Example 40.18 Mr. Singh purchases 10 computers @ Rs. 17,500 per computer. On each computer he earns Rs. 2000 and pays VAT @ 8%. What will be the total sale price of these 10 computers and how much VAT he has to pay?

Solution: Cost of one computer = Rs.17500

Value added (profit) = Rs. 2000

Total = Rs.19500

VAT(8%) = Rs.160

Selling price of one computer = Rs.19660

Total selling price of 10 computers = Rs.196600

Total VAT paid = Rs.1600

Example 40.19 A washing machine dealer, purchases 5 washing machines (WM) @Rs. 22,000 per unit and 2 WM @ 25,000 per unit from the company. After earning profit of Rs. 6000 on each machine. The dealer sells 5 WM at Rs. 28750 and 2 WM at Rs. 31750. How much percentage of VAT he has paid and what is the total amount paid by him to the government as VAT.

Solution: Cost of 5 Washing Machines = Rs.5 × 22000 = Rs.110000

Profit earned = 5 × 6000 = Rs. 30000

Total = Rs. 140000

Selling price = Rs. [5 × 28750] = Rs.143750

VAT Paid = Rs. (143750 – 140000) = Rs. 3750

VAT% $= \frac{3750}{30000} \times 100 = 12.5\%$

Cost of 2 Washing Machines = Rs. 50000

Profit added = Rs.12000

VAT (12.5%) = Rs. 1500

Total selling price = Rs. 63500

∴ Total VAT paid = Rs.(3750 + 1500)

= Rs. 5250

Example 40.20 Suppose a computer dealer sells computer at Rs. 12,240 and he purchases the same computer at Rs 8000. VAT levied on computers is @ 8% but he gets rebate @2% .Calculate how much VAT he has to pay and how much is the total collection of VAT by the government.

Solution: Cost Price = Rs. 8000, Selling Price = Rs.12240

$\therefore$ Profit + VAT $= \text{Rs. } 4240$

VAT(%) $= (8-2)\% = 6\%$

$$\therefore 4240 = \text{profit} + 6\% \text{ of profit} = x + \frac{6}{100}.x = \frac{53x}{50}$$

$$\text{or,} \quad x = Rs. \frac{4240 \times 50}{53} = Rs.\ 4000$$

$\therefore$ VAT paid $= \text{Rs. } (4240 - 4000) = \text{Rs. } 240$

Example 40.21 A dealer purchases dish washer (DW) at Rs. 15,000 and further sells it at Rs. 20,200. If VAT levied on DW is 4%, calculate profit earned by him and how much VAT he has to pay to the govt. Also calculate the total VAT given to the govt. in this whole transaction.

Solution: Cost Price of Dish washer = Rs.15000

Selling Price = Rs. 20200

$\therefore$ Difference = Rs. 5200

Let value added = Rs.x

$$\therefore \quad 5200 = x + 4\% \text{ of } x = \frac{26x}{25}$$

$$\text{or,} \quad x = 5200 \times \frac{25}{26} = Rs.5000$$

$\therefore$ Profit earned = Rs. 5000

VAT paid = Rs. 200

Example 40.22 Sushil purchases 100 Wall Clocks (WC) @ Rs. 70 per unit and he sold all these WC to Ramesh at Rs. 9250 where he earns profit of Rs. 2000. After adding value of Rs. 30 per unit Ramesh sells these WC in the market. If VAT is same on all these clocks, calculate how much VAT Sushil has to pay and at what price Ramesh sells these WC in the market.

Solution: Price paid by Sushil = Rs.(100 × 70) = Rs.7000

Profit earned = Rs.2000

Total = Rs.9000

Selling price (including VAT) = Rs. 9250

$\therefore$ VAT = Rs. 250

$$\therefore \text{VAT (\%)} = \frac{250}{2000} \times 100 = 12.5\%$$

Price paid by Ramesh = Rs. 9250

Value added = Rs. 3000

Total = Rs.12250

$$\text{VAT} = Rs. \frac{12.5}{100} \times 3000 = Rs.\ 375$$

Selling Price = Rs. (12250 + 375) = Rs.12625

CHECK YOUR PROGRESS 40.3

1. A wholesaler bought 2 quintals of rice at Rs. 4,000 per quintal on which he added value of Rs. 750 per quintal. If VAT levied is @ 8% then what will be its total sale price?
2. A wholesaler purchases wheat @ Rs. 1000 per quintal, and then after converting the wheat into flour he sells it to the retailer @ Rs. 15.20 per kg. If he pays VAT @4%, calculate the total profit earned by the wholesaler.
3. Sudhir a garment merchant purchases garments worth Rs. 50,000. By adding his profit of Rs. 15,000 he sold the whole stuff at Rs. 66,200. Calculate at which rate VAT was levied and total collection of VAT by the govt.
4. A manufacturing unit of AC (Air Conditioner) sold an AC to the dealer at certain rate who further sold it to a customer at Rs. 22,800 making a profit of 50%. If VAT is levied @ 4%, calculate the rate at which AC was sold by the manufacturing unit to the dealer.
5. Gopal Electronics purchases 50 T.V. sets @Rs.10,000 per set and earns Rs. 5,000 on each set as a profit. If the company pays Rs. 25000 to the govt. as VAT, calculate at what rate VAT is levied on T.V. set.
6. Bob Robert purchases 200 electric steam irons @ Rs. 750 each and he earns Rs. 25 on first 50 irons, Rs. 50 on next 50 irons, Rs. 75 on next 50 irons and Rs. 100 on rest 50 irons. If VAT is levied @ 8%, calculate total VAT paid by Bob Robert to the government.

40.4.5 Application of Sales Tax

A manufacturer produces goods worth Rs 100 and on that he has to pay 10% sales tax, which is Rs 10, then its total sale price is Rs 110,

Manufacturer: 100+10 =110

C.P+10%S.T =T.S.P (Total Selling Price)

Wholesaler purchases goods from the manufacturer at Rs 110 and adds Rs 20 as a profit and 10% sales tax, so his total sale price is Rs 143.

Wholesaler: 110+20 =130+13 =143

C.P+ Profit = S.P + 10%of S.P =T.S.P.

Retailer purchases the same commodity from the wholesaler at Rs 143 and adds Rs 27 as a profit which comes to Rs 170 + 10% sales tax. Now total sale price comes to be Rs 187.

Retailer: 143+27=170+17=187

C.P + Profit = S.P + 10%of S.P =T.S.P.

In the whole procedure total collection by the government in the form of sales tax is Rs. 40.

Government's total tax collection =10+13+17= 40.

40.4.6 Application of value added tax

Manufacturer= 100+10 =110

C.P+VAT=T.S.P

Wholesaler: $110+20 = 130$, but he has to pay tax on $130-110 = 20$ ie. Rs. 2.

$$\text{C.P}+\text{Profit} = \text{S.P}$$

$$130+2 = 132$$

$$\text{S.P}+\text{VAT} = \text{T.S.P}$$

Retailer: $132+27 = 159+2.7 = 161.7$

$$\text{C.P}+\text{Profit} = \text{S.P}+\text{VAT} = \text{T.S.P}$$

Government's total tax collection $= 10+2+2.7 = 14.7$

From the above illustration, it is clear that if sales tax and VAT are imposed on the goods whose cost price is same and same rate of taxes are imposed, in case of sales tax, Government collects Rs 40 but in the case of VAT the total collection by the Government is only Rs14.70

40.4.7 Difference between VAT and Sales Tax

SALES TAX	VAT
1. complex system.	1. simplified tax system.
2. different slabs of tax	2. only four slabs of tax
3. collected at one point i.e. first or last.	3. charged at each stage
4. no tax levied on value addition on subsequent sales	4. tax on each value addition
5. problems of multiple taxation	5. a set off is given for previous purchases
6. discouragement to disclosure	6. encouragement to disclosure

40.4.8 Merits of Indirect Taxes

1. Indirect taxes are usually hidden in the prices of goods and services being transacted and, therefore their presence is not felt so much.
2. If the indirect taxes are properly administered, the chances of tax evasion are less.
3. Indirect taxes are a powerful tool in moulding the production and investment activities of the economy i.e. they can guide the economy in its resource allocation.

40.4.9 Demerits of Indirect taxes

1. It is claimed and very rightly that these taxes negate the principle of ability-to-pay and are therefore unjust to the poor. Since one of the objectives is to collect enough revenue, they spread over to cover the items, which are purchased generally by the poor. This makes them regressive in effect.
2. If indirect taxes are heavily imposed on the luxury items then this will only help partially because taxing the luxuries alone will not yield adequate revenue for the State.
3. Direct taxes take away a part of the purchasing power of the taxpayer and that has the effect of reducing demand and prices. On the other hand, indirect taxes are added to the sale prices of the taxed goods without touching the purchasing power in the first place. The result is that in their case inflationary forces are fed through higher prices, higher costs and wages and again higher prices.

Example 40.23 A wholesaler purchases 15 meters of cloth from the manufacturer @Rs. 80 per meter and sells to the retailer after adding value of Rs. 20 per meter. The retailer sells the cloth and making a profit of Rs.50 per meter. Calculate how much total tax was paid to the government in the whole transaction, through (i) **VAT and (ii) Sales tax method**, considering that both taxes were levied @8%.

Solution :(i) Wholesaler's Cost Price = Rs.15 × 80 = Rs.1200
VAT (8%) = Rs. 96
Total = Rs.1296
Value added = Rs. 15 × 20 = Rs.300
VAT $= Rs.\ 300 \times \frac{8}{100} = Rs.\ 24$
Retailer's cost = Rs.1620
Value added by retailer = Rs.15 × 50 = Rs. 750
VAT $= Rs.\ \frac{8}{100} \times 750 = Rs.\ 60$
∴ Net selling price = Rs. 2430
Total Tax paid = Rs. [96 + 24 + 60] = Rs.180
(ii) Wholesaler's Cost Price = Rs.1200
Sales Tax = Rs. 96
Total = Rs.1296
Value added = Rs. 300
Total = Rs. 1596
Sales Tax $= Rs.\ (\frac{8}{100} \times 1596) = Rs.128$
Total = Rs.1724
Value added by retailer = Rs.15 × 50 = Rs.750
Total = Rs. (1724 + 750) = Rs. 2474
Sales Tax = 8% of 2474 = Rs. 198
Net Selling Price = Rs. 2672
Total Sales Tax paid = Rs.[96 + 128 + 198] = Rs. 422

Example 40.24 A manufacturer sold a TV set @Rs. 20,000 to the wholesaler. The wholesaler sells it to a retailer @Rs. 25500 and the retailer finally sells it to the customer @ Rs.31000. If VAT or sales tax whatever is levied is 10% extra at every stage, calculate the total tax collected by the government through (i) **VAT** and through (ii) **sales tax.**

Solution: (i) Wholesaler's cost Price = Rs. 20000
VAT (10%) = Rs. 2000
Total = Rs. 22000

Since, he sells at Rs. 25500, value added = Rs. 3500

$\therefore$ VAT (10%) = Rs. 350

$\therefore$ Cost of Retailer = Rs. 25850

Retailer sells at Rs. 31000. Therefore, Value added = Rs. 5150

$\therefore$ VAT (10%) = Rs. 515

Hence, total VAT paid = Rs.[2000 + 350 + 515]

= Rs. 2865

(ii) **When sales tax is paid**

Sales Tax by Manufactures = Rs.2000

Sales Tax by Wholesaler = Rs.25500 × 10% = Rs.2550

Sales Tax by retailer = Rs.31000 × 10% = Rs.3100

$\therefore$ Total Tax = Rs. [2000 + 2550 + 3100]

= Rs. 7650

Example 40.25 A firm produces 100 units of an item per day and sells all at the rate of Rs. 20 per unit to the wholesaler. If the Wholesaler added Rs. 500 as his profit and sells to retailer who adds Rs.1000 while selling, then calculate the total tax collected by the government, through (i) **VAT** and through (ii) **sales tax**, if both taxes are levied @10%.

Solution : (i) Cost of wholesaler = Rs. 20 × 100 = Rs.2000

VAT = Rs. 200

Total = Rs.2200

Value added (by wholesaler) = Rs. 500

VAT (10%) = Rs. 50

Total = Rs.2750

Value added by retailer = Rs.1000

VAT (10%) = Rs. 100

Total = Rs. 3850

$\therefore$ Total VAT (Tax) = Rs.[200 + 50 + 100]

= Rs. 350

(ii) **If sales tax is paid**

Cost of Wholesaler = Rs.2000

Sales Tax = Rs. 200

Total = Rs.2200

Value added (by Wholesaler) = Rs. 500

Total = Rs. 2700

$\therefore$ Tax = Rs. 270

$\therefore$ Total = Rs. 2970

Value added by retailer = Rs.1000

Total = Rs. 3970

Tax = Rs. 397

Total = Rs. 4367

Total Tax Paid = Rs.[200 + 270 + 397]

=Rs. 867

CHECK YOUR PROGRESS 40.4

1. A wholesaler purchases 15 chairs from the manufacturer @Rs.100 per chair excluding tax and sells them to a retailer after adding value of Rs.50 per chair. Calculate the total tax paid to the government in these transactions by (i) sales tax method, and (ii) by VAT method, if sales tax or VAT is levied @12.5% at each stage.
2. A dealer purchases 30 kgs of wheat @Rs.10 per kg plus VAT and after earning a profit of Rs.5 per kg the dealer sells it to the retailer. The retailer finally sells it to a customer @ Rs.22.55 per kg including VAT. Calculate how much tax is collected by the Government through VAT which is 10% at each stage.

LET US SUM UP

- Government has to perform many functions in the discharge of its duties, to meet these requirements they require capital. So, government collects money from the public in the form of fees, fines, surcharge and taxes.
- Taxes are the most important sources of revenue.
- The income of government through all sources is called public income or public revenue.
- Different tiers of government levies different taxes like, Central government levies-income tax, education cess, wealth tax, central excise and customs duty, central sales tax, etc, State government- Sales taxes (Now VAT), state excise duty, entertainment tax, agriculture revenue tax etc. Local bodies- property tax, professional tax, octroi, education cess, etc.
- Fines are compulsory payments, which are imposed to curb certain offences, and discipline people and fee is also compulsory payment, which are made when a person receives in return a particular benefit or services from the government. Whereas tax is legally compulsory payment levied by the government on the persons or companies to meet the expenditure incurred on conferring common benefits upon the people of a country.
- Direct taxes are those taxes whose burden cannot be shifted to others and the person who pays it to the government has to bear it. Indirect taxes are those whose burden can be shifted to others so that those who pay these taxes to the government do not bear the whole burden but pass it on wholly or partly to others.
- Excise duty can only be levied on those items which are manufactured in India (excluding goods produced or manufactured in special economic zones).
- Generally 16% of excise duty and 2% cess are imposed on most of the all goods, except few exceptions like in textile sector. In certain cases government can fix different tariff values for different classes.

- In case of delayed payment, interest should also be deposited at the rate of 13% p.m or Rs 1,000 per day for the period of delay after 5th or 15th as the case may be, whichever is higher, along with the duty.
- **Exemptions:** Central excise rules grant exemption from duty if goods are exported under bond, except exports to Nepal and Bhutan. Similarly, goods manufactured in special economic zones (SEZ) are not excisable goods and hence no excise duty can be levied on goods manufactured.
- Tax imposed by the government on the import and export of items (goods) is called customs duty.
- Tax paid by the consumer on the purchase of some items is called sales tax.
- VAT will replace the present sales tax in India. Under the current single-point system of tax levy, the manufacturer or importer of goods into a State is liable to sales tax. There is no sales tax on the further distribution channel. VAT, in simple terms, is a multi-point levy on each of the entities in the supply chain with the facility of set-off of input tax i.e. the tax paid at the stage of purchase of goods by a trader and on purchase of raw materials by a manufacturer.
- **RATES OF VAT:** There are four slabs of VAT imposed on the different goods, i.e. 1%, 4%, 12.5%, and 20%.
- **TAX CREDIT:** A dealer who is registered shall be entitled to a tax credit in respect of the turnover of purchases occurring during the tax period where the purchase arises in the course of his activities as a dealer and the goods are to be used by him directly or indirectly for the purpose of making sale.
- **NET TAX:** The net tax payable by a dealer for a tax period shall be determined by the formula:

  Net tax = $O - I - C$ where

  O = the amount of tax payable by the person at rates stipulated in respect of the taxable turnover arising in the tax period.

  I = the amount of the tax credit arising in the tax period to which the person is entitled for adjustment to the tax credit required by this Act.

  C = the amount if any brought forward from the previous tax period.
- If same rate of sales tax and VAT are imposed on the goods whose cost price is same then in case of sales tax, government collects more than in the case of VAT. However the coverage from VAT is more because in VAT there is very little chances of tax evasion.

SUPPORTIVE WEB SITES

- http://www.wikipedia.org
- http://www.dvat.gov.in

TERMINAL EXERCISE

1. A garments Company manufactures 20 quilts per day and uses cotton fiber (Rs. 100 per kg) and cotton cloth (Rs.50 per meter) as input. In making one quilt 2 kgs of cotton fibre and 5 meters of cloth are used. If excise duty on cotton fiber is 8% and on cloth it is 10% while on quilt it is 12%, calculate the total duty paid to the government when the value added by the Company is Rs.109 per quilt, if the transaction is (i) **without Cenvat** (ii) **with Cenvat**.
2. Mittal and Company produces steel utensils at the rate of 500 kgs per day after using inputs of Rs.25,000. The excise duty is 16% with 2% education cess. If the duty is paid 12 days after the due date, calculate the amount of penalty to be paid.
3. A computer dealer sells computer at Rs. 15,000 and he purchases the same computer at Rs 10,500. VAT levied on computers is @ 8% but he gets rebate @2% .Calculate how much VAT he has to pay and how much is the total collection of VAT by the government.
4. A wholesaler purchases 50 meters of cloth from the manufacturer @Rs. 80 per meter and sells to the retailer after adding value of Rs. 20 per meter. The retailer sells the cloth and making a profit of Rs. 50 per meter. Calculate how much total tax was paid to the government in the whole transaction, through (i) **VAT and** (ii) **Sales tax method,** considering that both taxes were levied @8%.

ANSWERS

CHECK YOUR PROGRESS 40.1

1. Rs. 12,826.26 2. Rs. 4500 3. Rs. 30,000

4. Rs. 1,92,000 5. Rs. 5,16,000

CHECK YOUR PROGRESS 40.2

1. Rs. 33,915 2. Rs. 913 3. Rs. 4590 Crores 4. Rs. 5,76,810

CHECK YOUR PROGRESS 40.3

1. Rs. 9,620 2. Rs. 500 3. 8% ; Rs. 1200

4. Rs. 15,000 5. 10% 6. Rs. 1,000

CHECK YOUR PROGRESS 40.4

1. (i) Rs. 493 (ii) Rs. 282 2. Rs. 61.50

TERMINAL EXERCISE

1. (i) Rs. 67,800 (ii) Rs. 40,248 2. Rs. 10,000

3. Rs. 255 (rounded off) 4. (i) Rs. 580 (ii) Rs. 1405

Annexure-A

List of Exempted Commodities

- Agricultural implements manually operated or animal driven.
- Aids and implements used by handicapped persons.
- Aquatic feed, poultry feed and cattle feed including grass, hay and straw.
- Betel leaves
- Books, Periodicals, Newspapers and Maps
- Charakha, Ambar Charakha, handlooms and handloom fabrics and Gandhi Topi.
- Charcoal
- Coarse grains other than paddy, rice and wheat
- Condoms and Contraceptives
- Cotton and silk yarn in hank.
- Curd, Lussi, butter milk and separated milk.
- Earthen pot
- Electricity energy
- Firewood
- Fishnet and fishnet fabrics
- Fresh milk and pasteurized milk
- Fresh plants, saplings and fresh flowers
- Fresh vegetables and fruits.
- Garlic and ginger
- Glass bangles
- Human blood and blood plasma
- Indigenous handmade musical instruments
- KUMKUM, BINDI ALTA AND SINDUR
- Meat, fish, prawn and other aquatic products when not cured or frozen; eggs and livestock and animal hair.
- National Flag.
- Organic manure
- Non-judicial stamp paper sold by Government Treasuries; postal items like envelope, post card etc. sold by Government; rupee note, when sold to the Reserve Bank of India and cheques, loose in book form.
- Raw wool.
- Semen including frozen semen.
- Silk worm laying, cocoon and raw silk.
- Slate and slate pencils
- Tender green coconut.
- Toddy, Neera and Arak.
- Unbranded bread
- Unprocessed and unbranded salt.

- Water other than
  a) aerated, mineral, distilled, medicinal, ionic, battery, de-mineralized water, and
  b) water sold in sealed container

List of Goods Taxed at 1%

- Bullion
- Articles of Gold, Silver and Precious Metals including Jewellery made from gold, silver and precious metals.
- Precious Stones and Semi-Precious Stones

List of Goods Taxed at 4%

- Agricultural implements not operated manually or not driven by animal
- All equipments for communications such as, Private Branch Exchange (PBX) and Electronic Private Automatic Branch Exchange (EPABX) etc.
- All intangible goods like copyright, patent, rep license, goodwill etc.
- All kinds of bricks including fly ash bricks, refractory bricks and asphaltic roofing, earthen tiles
- All types of yarn other than cotton and silk yarn in hank and sewing thread.
- Aluminum utensils and enameled utensils.
- Areca nut powder and betel nut
- Bamboo
- Bearings
- Beedi leaves
- Beltings
- Bicycles, tricycles, cycle rickshaws and parts
- Bitumen
- Bone meal
- Branded bread
- Bulk drugs
- Castings
- Centrifugal, monobloc and submersible pumps and parts thereof
- Coffee beans and seeds, cocoa pod, green tea leaf and chicory
- Chemical fertilizers, pesticides, weedicides and insecticides
- Coir and coir products excluding coir mattresses
- Cotton and cotton waste
- Crucibles
- Declared goods as specified in section 14 of the Central Sales Tax Act-1956
- Edible oils, oil cake and de-oiled cake
- Electrodes
- Exercise book, graph book and laboratory note book
- Ferrous and non-ferrous metals and alloys; non-metals such as aluminum, copper, zinc and extrusions of those
- Fibers of all types and fiber waste
- Flour, atta, Maida, suji, besan, etc.
- Fried grams
- Gur, jaggery, and edible variety of rub gur

- Hand pumps and spare parts
- Herb, bark, dry plant, dry root, commonly known as jari booti and dry flower
- Hose pipes
- Hosiery goods
- Husk and bran of cereals
- Ice
- Incense sticks commonly known as, agarbatti, dhupkathi or dhupbati
- Industrial cables (High voltage cables, ELPE Cables, jelly filled cables, optical fibres)
- IT products including computers, telephone and parts thereof, teleprinter and wireless equipment and parts thereof
- Kerosene oil sold through PDS
- Leaf plates and cups
- Murmuralu, pelalu, atukulu, puffed rice, muri
- Newars
- Napa Slabs (Rough flooring stones)
- Ores and minerals
- Paddy, rice, wheat and pulses
- Paper and newsprint
- Pipes of all varieties including GI pipes, CI pipes, ductile pipes and PVC pipes
- Plastic footwear
- Printed material including diary, calendar etc.
- Printing ink excluding toner and cartridges
- Processed and branded salt
- Pulp of bamboo, wood and paper
- Rail coaches engines and wagons
- Readymade garments
- Renewable energy devices and spare parts
- Safety matches
- Seeds
- Sewing machines
- Ship and other water vessels
- Silk fabrics
- Skimmed milk powder
- Solvent oils other than organic solvent oil
- Spices of all varieties and forms including cumin seed, aniseed, turmeric and dry chillies
- Sports goods excluding apparels and footwear
- Starch
- Sugar and khandsari
- Tamarind
- Tractors, threshers, harvesters and attachments and parts thereof
- Transmission towers
- Umbrella except garden umbrella
- Vanaspati (Hydrogenated vegetable oil)
- Vegetable oil including gingili oil and bran oil
- Writing instruments

List of Goods Taxed at 20%

- Petroleum Products other than liquid petroleum gas, Compressed Natural Gas and Kerosene such as
  - ✓ Naphtha
  - ✓ Aviation Turbine Fuel;
  - ✓ Spirit
  - ✓ Gasoline
  - ✓ Diesel(High Speed Diesel, Super Light Diesel Oil, Light Diesel Oil)
  - ✓ Furnace Oil
  - ✓ Organic Solvent
  - ✓ Coal Tar
  - ✓ Wax (except Petroleum wax used for manufacture of candles)
  - ✓ Mixture and combination of above products
- Liquor (Foreign and Indian Made Foreign Liquor).
- Country Liquor.
- Narcotics (Bhaang).
- Molasses.
- Rectified Spirit.
- Lottery Tickets.
- Brake Fluid.
- Food served in Star Hotels

41

APPLICATION OF CALCULUS IN COMMERCE AND ECONOMICS

We have learnt in calculus that when 'y' is a function of 'x', the derivative of y w.r.to x i.e. $\left(\frac{dy}{dx}\right)$ measures the instantaneous rate of change of y with respect to x. In Economics and commerce we come across many such variables where one variable is a function of the other. For example, the quantity demanded can be said to be a function of price. Supply and price or cost and quantity demanded are some other such variables. Calculus helps us in finding the rate at which one such quantity changes with respect to the other. Marginal analysis in Economics and Commerce is the most direct application of differential calculus. In this context, differential calculus also helps in solving problems of finding maximum profit or minimum cost etc., while integral calculus is used to find he cost function when the marginal cost is given and to find total revenue when marginal revenue is given.

In this lesson, we shall study about the total, average or marginal functions and the optimisation problems.

OBJECTIVES

After studying this lesson, you will be able to :

- define Total Cost, Variable Cost, Average Cost, Marginal Cost, Total Revenue, Marginal Revenue and Average Revenue;
- find marginal cost and average cost when total cost is given;
- find marginal revenue and average revenue when total revenue is given;
- find optimum profit and minimum total cost under given conditions; and
- find total cost/ total revenue when marginal cost/marginal revenue are given, under given conditions.

EXPECTED BACKGROUND KNOWLEDGE

- Derivative of a function
- Integration of a function

41.1 BASIC FUNCTIONS

Before studying the application of calculus, let us first define some functions which are used in business and economics.

41.1.1 Cost Function

The total cost C of producing and marketing x units of a product depends upon the number of units (x). So the function relating C and x is called Cost-function and is written as $C = C(x)$.

The total cost of producing x units of the product consists of two parts

(i) Fixed Cost

(ii) Variable Cost i.e. $C(x) = F + V(x)$

Fixed Cost : The fixed cost consists of all types of costs which do not change with the level of production. For example, the rent of the premises, the insurance, taxes, etc.

Variable Cost : The variable cost is the sum of all costs that are dependent on the level of production. For example, the cost of material, labour cost, cost of packaging, etc.

41.1.2 Demand Function

An equation that relates price per unit and quantity demanded at that price is called a demand function.

If 'p' is the price per unit of a certain product and x is the number of units demanded, then we can write the demand function as $x = f(p)$

or $p = g(x)$ i.e., price (p) expressed as a function of x.

41.1.3 Revenue function

If x is the number of units of certain product sold at a rate of Rs. 'p' per unit, then the amount derived from the sale of x units of a product is the total revenue. Thus, if R represents the total revenue from x units of the product at the rate of Rs. 'p' per unit then

$$R = p.x \text{ is the total revenue}$$

Thus, the Revenue function $R(x) = p.x. = x.p(x)$

41.1.4 Profit Function

The profit is calculated by subtracting the total cost from the total revenue obtained by selling x units of a product. Thus, if P (x) is the profit function, then

$$P(x) = R(x) - C(x)$$

41.1.5 Break-Even Point

Break even point is that value of x (number of units of the product sold) for which there is no profit or loss.

i.e. At Break-Even point $P(x) = 0$

or $R(x) - C(x) = 0$ i.e. $R(x) = C(x)$

Let us take some examples.

Example 41.1 For a new product, a manufacturer spends Rs. 1,00,000 on the infrastructure and the variable cost is estimated as Rs.150 per unit of the product. The sale price per unit was fixed at Rs.200.

Find (i) Cost function (ii) Revenue function

(iii) Profit function, and (iv) the break even point.

Solution : (i) Let x be the number of units produced and sold,

then cost function $C(x) =$ Fixed cost + Variable Cost

$$= 1{,}00{,}000 + 150x$$

(ii) Revenue function $= p.x = 200x$

(iii) Profit function $P(x) = R(x) - C(x)$

$$= 200x - (100{,}000 + 150x)$$
$$= 50x - 100{,}000$$

(iv) At Break-Even point $P(x) = 0$

$$50x - 100{,}000 = 0$$

$$x = \frac{100{,}000}{50} = 2000$$

Hence $x = 2000$ is the break even point.

i.e. When 2000 units of the product are produced and sold, there will be no profit or loss.

Example 41.2 A Company produced a product with Rs 18000 as fixed costs. The variable cost is estimated to be 30% of the total revenue when it is sold at a rate of Rs. 20 per unit. Find the total revenue, total cost and profit functions.

Solution : (i) Here, price per unit (p) = Rs. 20

Total Revenue $R(x) = p.x = 20x$ where x is the number of units sold.

(ii) Cost function $C(x) = 18000 + \frac{30}{100}R(x)$

$$= 18000 + \frac{30}{100} \times 20x$$

$$= 18000 + 6x$$

(iii) Profit function $P(x) = R(x) - C(x)$

$$= 20x - (18000 + 6x)$$

$$= 14x - 18000$$

Example 41.3 A manufacturing company finds that the daily cost of producing x items of a product is given by $C(x) = 210x + 7000$

(i) If each item is sold for Rs. 350, find the minimum number that must be produced and sold daily to ensure no loss.

(ii) If the selling price is increased by Rs. 35 per piece, what would be the break even point.

Solution :

(i) Here, $R(x) = 350x$ and $C(x) = 210x + 7000$

$\therefore \quad P(x) = 350x - 210x - 7000$

$$= 140x - 7000$$

For no loss $\quad P(x) = 0$

$\Rightarrow \quad 140x - 7000 = 0$ or $x = 50$

Hence, to ensure no loss, the company must produce and sell at least 50 items daily.

(ii) When selling price is increased by Rs. 35 per unit,

$$R(x) = (350 + 35)x = 385x$$

$\therefore \quad P(x) = 385x - (210x + 7000)$

$$= 175x - 7000$$

At Break even point $\quad P(x) = 0$

$\Rightarrow \quad 175x - 7000 = 0$

$\therefore \quad x = \dfrac{7000}{175} = 40$

CHECK YOUR PROGRESS 41.1

1. The fixed cost of a new product is Rs. 18000 and the variable cost per unit is Rs. 550. If demand function $p(x) = 4000 - 150x$, find the break even values.
2. A company spends Rs. 25000 on infrastructure and the variable cost of producing one item is Rs. 45. If this item is sold for Rs. 65, find the break-even point.
3. A television manufacturer find that the total cost of producing and selling x television sets is $C(x) = 50x^2 + 3000x + 43750$. Each product is sold for Rs. 6000. Determine the break even points.
4. A company sells its product at Rs.60 per unit. Fixed cost for the company is Rs.18000 and the variable cost is estimated to be 25 % of total revenue. Determine :

   (i) the total revenue function (ii) the total cost function (iii) the breakeven point.

5. A profit making company wants to launch a new product. It observes that the fixed cost of the new product is Rs. 35000 and the variable cost per unit is Rs. 500. The revenue function for the sale of x units is given by $R(x)=5000x-100x^2$. Find :

(i) Profit function (ii) break even values (iii) the values of x that result in a loss.

41.1.6 Average and Marginal Functions

If two quantities x and y are related as y = f (x), then the average function may be defined as $\frac{f(x)}{x}$ and the marginal function is the instantaneous rate of change of y with respect to x. i.e.

Marginal function is $\frac{dy}{dx}$ or $\frac{d}{dx}(f(x))$

Average Cost : Let $C = C(x)$ be the total cost of producing and selling x units of a product, then the average cost (AC) is defined as $AC = \frac{C}{x}$

Thus, the average cost represents per unit cost.

Marginal Cost : Let C = C(x) be the total cost of producing x units of a product, then the marginal cost (MC), is defined to be the rale of change of C (x) with respect to x. Thus

$$MC = \frac{dC}{dx} \text{ or } \frac{d}{dx}(C(x)).$$

Marginal cost is interpreted as the approximate cost of one additional unit of output.

For example, if the cost function is $C = 0.2x^2 + 5$, then the marginal cost is $MC = 0.4x$

∴ The marginal cost when 5 units are produced is

$$[MC]_{x=5} = (0.4)(5) = 2$$

i.e. when production is increased from 5 units to 6, then the cost of additional unit is approximately Rs. 2.

However, the actual cost of producing one more unit after 5 units is C(6) – C(5) = Rs. 2.2

Example 41.4 The cost function of a firm is given by $C=2x^2+x-5$.

Find (i) the average cost (ii) the marginal cost, when x = 4

Solution :

(i)
$$AC = \frac{C}{x} = \frac{2x^2+x-5}{x}$$

$$= 2x+1-\frac{5}{x}$$

At $$x=4,\ AC=2(4)+1-\frac{5}{4}$$

$$=9-1.25=7.75$$

(ii) $$MC=\frac{d}{dx}(C)=4x+1$$

$\therefore$ MC at $x=4=4(4)+1=16+1=17$

Example 41.5 Show that he slope of average cost curve is equal to $\frac{1}{x}(MC-AC)$ for the total cost function $C=ax^3+bx^2+cx+d$.

Solution : Cost function $C=ax^3+bx^2+cx+d$

Average cost $$AC=\frac{C}{x}=ax^2+bx+c+\frac{d}{x}$$

Marginal cost $$MC=\frac{d}{dx}(C)=3ax^2+2bx+c$$

Slope of AC curve $$=\frac{d}{dx}(AC)=\frac{d}{dx}\left(ax^2+bx+c+\frac{d}{x}\right)$$

$$=2ax+b-\frac{d}{x^2}$$

$\therefore$ slope of AC curve $$=2ax+b-\frac{d}{x^2}$$

$$=\frac{1}{x}\left[2ax^2+bx-\frac{d}{x}\right]$$

slope of AC curve $$=\frac{1}{x}\left[\left(3ax^2+2bx+c\right)-\left(ax^2+bx+c+\frac{d}{x}\right)\right]$$

$$=\frac{1}{x}[MC-AC]$$

Example 41.6 If the total cost function C of a product is given by

$$C=2x\left(\frac{x+7}{x+5}\right)+7$$

Prove that he marginal cost falls continuously as the output increases.

Solution : Here $C=2x\left(\frac{x+7}{x+5}\right)+7=2\left(\frac{x^2+7x}{x+5}\right)+7$

$$\therefore \quad \frac{dC}{dx} = 2\left[\frac{(x+5)(2x+7)-\left(x^2+7x\right).1}{(x+5)^2}\right]$$

$$= 2\left[\frac{2x^2+17x+35-x^2-7x}{(x+5)^2}\right]$$

$$= 2\left[\frac{x^2+10x+35}{(x+5)^2}\right]$$

$$= 2\left[\frac{(x+5)^2+10}{(x+5)^2}\right]$$

$$= 2\left[1+\frac{10}{(x+5)^2}\right]$$

$$\therefore \quad MC = 2\left[1+\frac{10}{(x+5)^2}\right]$$

It is clear that when x increases $(x+5)^2$ increases and so $\frac{10}{(x+5)^2}$ decreases and hence MC decreases.

Thus, the marginal cost falls continuously as the output increases.

41.2 AVERAGE REVENUE AND MARGINAL REVENUE

We have already learnt that total revenue is the total amount received by selling x items of the product at a price 'p' per unit. Thus, $R = p.x$

Average Revenue : If R is the revenue obtained by selling x units of the product at a price 'p' per unit, then the term average revenue means the revenue per unit, and is written as AR.

$$\therefore \quad AR = \frac{R}{x}$$

But $\quad R = p.\ x.$

$$\therefore \quad AR = \frac{p.x}{x} = p$$

Hence, average revenue is the same as price per unit.

Marginal Revenue : The marginal revenue (MR) is defined as the rate of change of total revenue with respect to the quantity demanded.

$$\therefore \qquad MR = \frac{d}{dx}(R) \text{ or } \frac{dR}{dx}$$

The marginal revenue is interpreted as the approximate revenue received from producing and selling one additional unit of the product.

Example 41.7 The total revenue received from the sale of x units of a product is given by $R(x) = 12x + 2x^2 + 6$.

Find (i) the average revenue

(ii) the marginal revenue

(iii) marginal revenue at $x = 50$

(iv) the actual revenue from selling 51st item

Solution : (1) Average revenue $AR = \frac{R}{x} = \frac{12x + 2x^2 + 6}{x}$

$$= 12 + 2x + \frac{6}{x}$$

(ii) Marginal revenue $\qquad MR = \frac{d}{dx}(R) = 12 + 4x$.

(iii) $\qquad [MR]_{x=50} = 12 + 4(50) = 212$

(iv) The actual revenue received on selling 51st item

$$= R(51) - R(50)$$

$$= \left[12(51) + 2(51)^2 + 6\right] - \left[12(50) + 2(50)^2 + 6\right]$$

$$= 12(51 - 50) + 2\left[51^2 - 50^2\right]$$

$$= 12 + 2 \times 101$$

$$= 12 + 202 = 214$$

Example 41.8 The demand function of a product for a manufacturer is $p(x) = ax + b$

He knows that he can sell 1250 units when the price is Rs.5 per unit and he can sell 1500 units at a price of Rs.4 per unit. Find the total, average and marginal revenue functions. Also find the price per unit when the marginal revenue is zero.

Solution : Here, $\qquad p(x) = ax + b$

and when $\qquad x = 1250,\ p = \text{Rs } 5$

$\therefore \qquad 5 = 1250\,a + b \qquad$ ....(i)

and when $\qquad x = 1500,\ p = \text{Rs } 4$

$\therefore \qquad 4 = 1500\,a + b \qquad$ ....(ii)

Solving (i) and (ii) we get $a=-\frac{1}{250}, b=10$

$\therefore$ Demand function is expressed as $p(x)=10-\frac{x}{250}$

$\therefore$ Total Revenue $\qquad R=p.x=10x-\frac{x^2}{250}$

Average Revenue $\qquad p=10-\frac{x}{250}$

and Marginal revenue $\qquad MR=10-\frac{2x}{250}=10-\frac{x}{125}$

Now, when MR = 0 we have $10-\frac{x}{125}=0 \therefore x=1250$

Thus, $\qquad p=10-\frac{1250}{250}=5$

i.e., price per unit is Rs. 5.

Thus, at a price of Rs 5 per unit the marginal revenue vanishes.

Example 41.9 The demand function of a monopolist is given by $p = 1500 - 2x - x^2$. Find: (i) the revenue function, (ii) the marginal revenue function (iii) the MR when $x=20$

Solution : $p = 1500 - 2x - x^2$

(i) $\therefore$ Revenue function $\qquad R = p.\ x = 1500x - 2x^2 - x^3$

(ii) Marginal revenue $\qquad MR=\frac{d}{dx}(R)=1500-4x-3x^2$

(iii) $\qquad [MR]_{x=20}=1500-80-1200=220$

Note : In the absence of any competition, the business is said to be operated as monopoly business, and the businessman is said to be a monopolist. Thus, in case of a monopolist, the price of the product depends upon the number of units produced and sold.

CHECK YOUR PROGRESS 41.2

1. The total cost C(x) of a company is given as $C(x) = 1000 + 25x + 2x^2$ where x is the output. Determine:

   (i) the average cost (ii) the marginal cost (iii) the marginal cost when 15 units are produced, and (iv) the actual cost of producing 15th unit.

2. The cost function of a firm is given by $C = 2x^2 + 3x + 4$.

   Find (i) The average cost (ii) the marginal cost, (iii) Marginal cost, when $x = 5$.

3. The total cost function of a firm is given as

$$C(x) = 0.002x^3 - 0.04x^2 + 5x + 1500$$

   where x is the output. Determine :

   (i) the average cost (ii) the marginal average cost (MAC)

   (iii) the marginal cost (iv) the rate of change of MC with respect to x

4. The average cost function (AC) for a product is given by

   $AC = 0.006x^2 - 0.02x - 30 + \frac{5000}{x}$, where x is the output. Find (i) the marginal cost function (ii) the marginal cost when 50 units are produced.

5. The total cost function for a company is given by $C(x) = \frac{3}{4}x^2 - 7x + 27$. Find the level of output for which MC = AC.

6. The demand function for a monopolist is given by $x = 100 - 4p$, where x is the number of units of product produced and sold and p is the price per unit.

   Find : (i) total revenue function (ii) average revenue function (iii) marginal revenue function and (iv) price and quantity at which MR = 0.

7. A firm knows that the demand function for one of its products is linear. It also knows that it can sell 1000 units when the price is Rs.4 per unit and it can sell 1500 units when the price is Rs.2 per unit.

   Determine :

   (i) the demand function (ii) the total revenue function (iii) the average revenue function (iv) the marginal revenue function.

8. The demand function for a product is given by $p = \frac{5}{x+3}$. Show that the marginal revenue function is a decreasing function.

Minimization of Average cost or total cost and Maximization of total revenue, the total profit.

We know that if $C = C(x)$ is the total cost function for x units of a product, then the average cost (AC) is given by

$$AC = \frac{C(x)}{x}$$

In Economics and Commerce, it is very important to find the level of output for which the average cost is minimum. Using calculus, this can be calculated by solving $\frac{d}{dx}(AC) = 0$ and to get that value of x for which $\frac{d^2}{dx^2}(AC) > 0$.

Similarly, when we are interested to find the level of output for which the total revenue is maximum. we solve $\frac{d}{dx}[R(x)] = 0$ and find that value of x for which $\frac{d^2}{dx^2}[R(x)] < 0$.

Similarly we can find the value of x for maximum profit by solving $\frac{d}{dx}[P(x)] = 0$ and to find that value of x for which $\frac{d^2}{dx^2}[P(x)] < 0$.

Example 41.10 The manufacturing cost of an item consists of Rs.6000 as over heads, material cost Rs. 5 per unit and labour cost Rs. $\frac{x^2}{60}$ for x units produced. Find how many units must be produced so that the average cost is minimum.

Solution : Total cost $C(x) = 6000 + 5x + \frac{x^2}{60}$

$$\therefore \quad AC = \frac{6000}{x} + 5 + \frac{x}{60}$$

$$\text{Now.} \quad \frac{d}{dx}(AC) = -\frac{6000}{x^2} + \frac{1}{60}$$

$$\therefore \quad \frac{d}{dx}(AC) = 0 \quad \Rightarrow \quad -\frac{6000}{x^2} + \frac{1}{60} = 0$$

$$\Rightarrow \quad x^2 = 3600,00$$

$$\Rightarrow \quad x = 600$$

$$\frac{d^2}{dx^2}(AC) = +\frac{12000}{x^3} > 0 \text{ at } x = 600$$

Hence AC is minimum when $x = 600$

Example 41.11 The total cost function of a product is given by

$$C(x) = x^3 - \frac{615x^2}{2} + 15750\,x + 18000,$$

where x is the number of units produced. Determine the number of units that must be produced to minimize the total cost.

Solution : We have, $C(x) = x^3 - \frac{615x^2}{2} + 15750x + 18000$

$$\therefore \quad \frac{d}{dx}[C(x)] = 3x^2 - 615x + 15750$$

$$\frac{d}{dx}[C(x)] = 0$$

$\Rightarrow$ $3x^2 - 615x + 15750 = 0$ or, $x^2 - 205x + 5250 = 0$

or, $(x-175)(x-30) = 0$

This gives $x = 175, x = 30$

$$\frac{d^2}{dx^2}[C(x)] = 6x - 615 \text{, which is positive at } x = 175$$

So, C(x) is minimum when 175 units are produced.

Example 41.12 The demand function for a manufacturer's product is $x = 70 - 5p$, where x is the number of units and 'p' is the price per unit. At what value of x will there be maximum revenue ? What is the maximum revenue ?

Solution : Demand function is given as $x = 70 - 5p$

This gives, $p = \frac{70 - x}{5}$

$\therefore$ $R(x) = p.x = \frac{70x - x^2}{5}$

$$\frac{d}{dx}[R(x)] = \frac{1}{5}[70 - 2x]$$

$\frac{d}{dx}[R(x)] = 0$ Gives $x = 35$

Now, $\frac{d^2}{dx^2}[R(x)] = \frac{1}{5}(-2) < 0$

$\therefore$ for maximum revenue, $x = 35$

and Maximum revenue $= \text{Rs.} \frac{70(35) - (35)^2}{5} = \text{Rs. } 245$

Example 41.13 A company charges Rs.700 for a radio set on an order of 60 or less sets. The charge is reduced by Rs.10 per set for each set ordered in excess of 60. Find the largest size order company should allow so as to receive a maximum revenue.

Solution : Let x be the number of sets ordered in excess of 60.

i.e. number of sets ordered $= (60 + x)$

$\therefore$ Price per set = Rs. $(700 - 10x)$

$\therefore$ Total revenue $R = (60 + x)(700 - 10x)$

$$\frac{dR}{dx}=(60+x)(-10)+(700-10x)\cdot 1$$

$$=-600-10x+700-10x$$

$$=100-20x$$

$$\frac{dR}{dx}=0 \Rightarrow x=5$$

$$\frac{d^2R}{dx^2}=-20<0$$

∴ For maximum revenue, the largest size of order

$$=(60+5)\text{ sets}=65\text{ sets}$$

Example 41.14 The cost function for x units of a product produced and sold by a company is $C(x)=250+0.005x^2$ and the total revenue is given as $R=4x$. Find how many items should be produced to maximize the profit. What is the maximum profit ?

Solution : $C(x)=250+0.005x^2$ and $R(x)=4x$

∴ Profit function $\quad P(x)=R(x)-C(x)$

$$=4x-250-0.005x^2$$

$$\frac{d}{dx}[P(x)]=4-0.010x$$

∴ $\quad \frac{d}{dx}[P(x)]=0$ gives

$$4-0.01x=0 \qquad \text{or} \qquad x=\frac{4}{.01}=400$$

and $\quad \frac{d^2[P(x)]}{dx^2}=-0.01<0$

∴ For maximum profit, x = 400 and

$$\text{maximum profit} = \text{Rs.}\left[4(400)-250-\frac{5}{1000}\times 400\times 400\right]$$

$$=\text{Rs.}[1600-250-800]$$

$$=\text{Rs. }550$$

Example 41.15 A firm has found from past experience that its profit in terms of number of units x produced, is given by

$$P(x)=-\frac{x^3}{3}+729x+2700,\ 0\le x\le 35.$$

Compute :

(i) the value of x that maximizes the profit, and

(ii) the profit per unit of the product, when this maximum level is achieved.

Solution : $P(x) = -\frac{x^3}{3} + 729x + 2700$

$$\therefore \quad \frac{d}{dx}[P(x)] = -x^2 + 729$$

$$\therefore \quad \frac{d}{dx}[P(x)] = 0 \Rightarrow x = 27$$

$$\frac{d^2}{dx^2}[P(x)] = -2x < 0$$

$\therefore$ For maximum profit , $x = 27$

(ii) $$P(x) = -\frac{x^3}{3} + 729x + 2700$$

$$\therefore \text{ Profit per unit } = -\frac{x^2}{3} + 729 + \frac{2700}{x}$$

$$= \text{Rs.}\left[-\frac{729}{3} + 729 + \frac{2700}{27}\right]$$

$$= \text{Rs.}[-243 + 729 + 100]$$

$$= \text{Rs.}[829 - 243 = \text{Rs. } 586]$$

CHECK YOUR PROGRESS 41.3

1. The cost of manufacturing an item consists of Rs.3000 as over heads, material cost Rs. 8 per item and the labour cost $\frac{x^2}{30}$ for x items produced. Find how many items must be produced to have the average cost as minimum.

2. The cost function for a firm is given by

   $C = 300x - 10x^2 + \frac{1}{3}x^3$, where x is the output.

   Determine :

   (i) the output at which marginal cost is minimum,

   (ii) the output at which average cost is minimum, and

   (iii) the output at which average cost is equal to the marginal cost.

3. If $C = 0.01x^2 + 5x + 100$ is the cost function for x items of a product. At what level of production ,x, is there minimum average cost ? What is this minimum average cost ?

4. A television manufacturer produces x sets per week so that the total cost of production is given by the relation $C(x) = x^3 - 195x^2 + 6600x + 15000$. Find how many television sets must be manufactured per week to minimize the total cost.

5. The demand function for a product marketed by a company is $p = \frac{80 - x}{4}$, where x is the number of units and p is the price per unit. At what value of x will there be maximum revenue? What is this maximum revenue ?

6. A company charges Rs. 15000 for a refrigerator on orders of 20 or less refrigerators. The charge is reduced on every set by Rs.100 per piece for each piece ordered in excess of 20. Find the largest size order the company should allow so as to receive a maximum revenue.

7. A firm has the following demand and the average cost-functions:

$$x = 480 - 20p \qquad \text{and} \quad AC = 10 + \frac{x}{15}$$

Determine the profit maximizing output and price of the monopolist.

8. A given product can be manufactured at a total cost $C = \text{Rs.}\left(\frac{x^2}{100} + 100x + 40\right)$, where x is the number of units produced. If $p = \text{Rs.}\left(200 - \frac{x}{400}\right)$ is the price at which each unit can be sold, then determine x for maximum profit.

41.3 APPLICATION OF INTEGRATION TO COMMERCE AND ECONOMICS

We know that marginal function is obtained by differentiating the total function. Now, when Marginal function is given and initial values are given, then total function can be obtained with the help of integration.

41.3.1 Determination of cost function

If C denotes the total cost and $MC = \frac{dC}{dx}$ is the marginal cost, then we can write

$C = C(x) = \int (MC)\, dx + k$, where k is the constant of integration, k, being the constant, is the fixed cost.

Example 41.16 The marginal cost function of manufacturing x units of a product is $5 + 16x - 3x^2$. The total cost of producing 5 items is Rs. 500. Find the total cost function.

Solution : Given, $\quad MC = 5 + 16x - 3x^2$

$$\therefore \quad C(x) = \int \left(5 + 16x - 3x^2\right) dx$$

Notes

$$= 5x + 16\cdot\frac{x^2}{2} - 3\cdot\frac{x^3}{3} + k$$

$$C(x) = 5x + 8x^2 - x^3 + k$$

When $x = 5, C(x) = C(5) = \text{Rs. } 500$

or, $$500 = 25 + 200 - 125 + k$$

This gives, $$k = 400$$

$\therefore$ $$C(x) = 5x + 8x^2 - x^3 + 400$$

Example 41.17 The marginal cost function of producing x units of a product is given by $MC = \frac{x}{\sqrt{x^2+2500}}$. Find the total cost function and the average cost function if the fixed cost is Rs. 1000.

Solution : $$MC = \frac{x}{\sqrt{x^2+2500}}$$

$\therefore$ $$C(x) = \int \frac{x}{\sqrt{x^2+2500}}dx + k$$

Let $$x^2 + 2500 = t^2 \Rightarrow x\,dx = t\,dt$$

$\therefore$ $$C(x) = \int \frac{tdt}{t} + k$$

$$C(x) = \int dt + k = t + k = \sqrt{x^2+2500} + k$$

When $x = 0$, $C(0) = \text{Rs } 1000$

$\therefore$ $$1000 = \sqrt{2500} + k = 50 + k$$

or, $$k = 950$$

$\therefore$ $$C(x) = \sqrt{x^2+2500} + 950$$

$$AC = \sqrt{1 + \frac{2500}{x^2}} + \frac{950}{x}$$

Example 41.18 The marginal cost (MC) of a product is given to be a constant multiple of number of units (x) produced. Find the total cost function, if fixed cost is Rs.5000 and the cost of producing 50 units is Rs. 5625.

Solution : Here $MC \propto x$ i.e $MC = k_1 x$ (k_1 is a constant)

$\therefore$ $$\frac{dC}{dx} = k_1 x \Rightarrow C = \int k_1 x dx + k_2$$

$$\therefore \quad C = k_1\frac{x^2}{2} + k_2$$

since fixed cost = Rs 5000 $\therefore x = 0 \Rightarrow C = 5000$

$$\Rightarrow \quad k_2 = 5000$$

Now cost of producing 50 units is Rs 5625

$$\therefore \quad 5625 = k_1\frac{2500}{2} + 5000$$

$$\Rightarrow \quad 625 = 1250\,k_1 \Rightarrow k_1 = \frac{1}{2}$$

Hence $C = \frac{x^2}{4} + 5000$, is the required cost function.

41.3.2 Determination of Total Revenue Function

If R(x) denotes the total revenue function and MR is the marginal revenue function, then

$$MR = \frac{d}{dx}[R(x)]$$

$\therefore \quad R(x) = \int (MR)\,dx + k$ Where k is the constant of integration.

Also, when R (x) is known, the demand function can be found as $p = \frac{R(x)}{x}$

Example 41.19 The marginal revenue function of a commodity is given as

$MR = 12 - 3x^2 + 4x$. Find the total revenue and the corresponding demand function.

Solution : $MR = 12 - 3x^2 + 4x$

$$\therefore \quad R = \int\left(12 - 3x^2 + 4x\right)dx + k$$

$R = 12x - x^3 + 2x^2$ [constant of integration is zero in this case]

$\therefore$ Revenue function is given by $R = 12x + 2x^2 - x^3$

Since $x = 0, R = 0 \Rightarrow k = 0$

$\therefore \quad p = \frac{R}{x} = 12 + 2x - x^2$ is the demand function.

Example 41.20 The marginal revenue function for a product is given by

$$MR = \frac{6}{(x-3)^2} - 4$$

Find the total revenue function and the demand function.

Solution : $MR = \frac{6}{\left(x-3^2\right)} - 4$

$$\therefore \quad R = \int \left[\frac{6}{(x-3)^2} - 4\right] dx = -\frac{6}{x-3} - 4x + k$$

$x = 0, R = 0 \Rightarrow k = -2$

$$\therefore \quad R = -\frac{6}{x-3} - 4x - 2,$$ which is the required revenue function.

Now, $$p = \frac{R}{x} = -\frac{6}{x(x-3)} - 4 - \frac{2}{x}$$

$$= -\frac{6}{x(x-3)} - \frac{2}{x} - 4$$

$$= \frac{-6-2x+6}{x(x-3)} - 4$$

$$= \frac{-2}{x-3} - 4 = \frac{2}{3-x} - 4$$

$\therefore$ The demand function is given by $p = \frac{2}{3-x} - 4$.

CHECK YOUR PROGRESS 41.4

1. The marginal cost of production is $MC = 20 - 0.04x + 0.003x^2$, where x is the number of units produced. The fixed cost is Rs. 7000. Find the total cost and the average cost function.
2. The marginal cost function of manufacturing x units of a product is given by . $MC = 3x^2 - 10x + 3$. The total cost of producing one unit of the product is Rs.7. Find the total and average cost functions.
3. The marginal cost function of a commodity is given by $MC = \frac{14000}{\sqrt{7x+4}}$ and the fixed cost is Rs. 18000. Find the total cost and average cost of producing 3 units of the product.
4. If the marginal revenue function is $MR = \frac{4}{(2x+3)^2} - 1$, find the total revenue and the demand function.

5. If $MR = 20 - 5x + 3x^2$, find total revenue function.

6. If $MR = 14 - 6x + 9x^2$, find the demand function.

LET US SUM UP

Notes

- Cost function of producing and selling x units of a product depends upon x.
- $C(x) =$ Fixed cost + Variable cost.
- Demand function written as $p = f(x)$ or $x = f(p)$ where p is the price per unit, and x number of units produced.
- Revenue function, is the money derived from sale of x units of a product. $\therefore$ $R(x) = p.x.$
- Profit function $= R(x) - C(x)$. i.e. $P(x) = R(x) - C(x)$
- Break even point is that value of x for which $P(x) = 0$
- Average cost $AC = \frac{C}{x}$
- Marginal cost $MC = \frac{d}{dx}[C(x)]$
- Average revenue $AR = \frac{R}{x} = p$
- Marginal revenue $= MR = \frac{d}{dx}(R)$
- For minimization of AC, solve $\frac{d}{dx}(AR) = 0$ and then find that value of x for which $\frac{d^2}{dx^2}(AR) > 0$.
- For maximization of R(x) or P(X), solve $\frac{d}{dx}(R(x)) = 0$ or $\frac{d}{dx}(P(x)) = 0$ and then find x for which 2nd order derivative is negative.
- $C(x) = \int (MC)dx + k_1$
- $R(x) = \int (MR)dx + k_2$

SUPPORTIVE WEB SITES

http://www.wikipedia.org
http://mathworld.wolfram.com

Notes

TERMINAL EXERCISE

1. A profit making company wants to launch a new product. It observes that the fixed cost of the new product is Rs.7500 and the variable cost is Rs.500. The revenue received on sale of x units is $2500\,x - 100\,x^2$. Find : (i) profit function (ii) break even point.

2. A company paid Rs. 16100 towards rent of the building and interest on loan. The cost of producing one unit of a product is Rs. 20. If each unit is sold for Rs. 27, find the break even point.

3. A company has fixed cost of Rs. 26000 and the cost of producing one unit is Rs.30. If each unit sells for Rs. 43, find the breakeven point.

4. A company sells its product for Rs. 10 per unit. Fixed costs for the company are Rs. 35000 and the variable costs are estimated to run 30 % of total revenue. Determine : (i) the total revenue function (ii) total cost function and (iii) quantity the company must sell to cover the fixed cost.

5. The fixed cost of a new product is Rs. 30000 and the variable cost per unit is Rs. 800. If the demand function is $p\,(x) = 4500 - 100\,x$ find the break even values.

6. If the total cost function C of a product is given by

   $C = 3x\left(\dfrac{x+7}{x+5}\right)$. Prove that the marginal cost falls continuously as the output increases.

7. The average cost function (AC) for a product is given by $AC = x + 5 + \dfrac{36}{x}$, where x is the output. Find the output for which AC is increasing and the output for which AC is decreasing with increasing output. Also find the total cost C and the marginal cost MC.

8. A firm knows that the demand function for one of its products in linear. It also knows that it can sell 1000 units when the price is Rs. 4 per unit, and it can sell 1500 units when the price is Rs. 2 per unit.

   Find : (i) the demand function (ii) the total revenue function (iii) the average revenue function and (iv) the marginal revenue function.

9. The average cost function AC for a product is given by $AC = x + 5 + \dfrac{36}{x}, x \neq 0$. Find the total cost and marginal cost functions. Also find MC when $x = 10$.

10. The demand function for a product is given as $p = 30 + 2x - 5x^2$, where x is the number of units demanded and p is the price per unit. Find (i) Total revenue (ii) Marginal revenue (iii) MR when $x = 3$.

11. For the demand function $p = \dfrac{-5}{3+x}$, show that the marginal revenue function is an increasing function.

12. The demand function for a product is given as $3x = 24 - 2p$, where x is the number of units demanded at a price of p per unit.

    Find : (i) the Revenue function R in terms of p (ii) the price and number of units demanded for which revenue is maximum.

13. The cost function C of a firm is given as

$$C = 100x - 10x^2 + \frac{1}{3}x^3,$$

    Calculate : (i) output, at which the marginal cost is minimum. (ii) output, at which the average cost is minimum. (iii) output, at which the average cost is equal to the marginal cost.

14. The profit of a monopolist is given by $p(x) = \frac{8000x}{500+x} - x$. Find the value of x for which the p (x) is maximum. Also find the maximum profit.

15. The marginal cost of producing x units of a product is given by $MC = x\sqrt{x+1}$. The cost of producing 3 units is Rs. 7800. Find the cost function.

16. The marginal revenue function for a firm is given by

$$MR = \frac{2}{x+3} - \frac{2x}{(x+3)^2} + 5.$$

    Show that the demand function is $p = \frac{2}{x+3} + 5$.

17. The cost function of producing x units of a product is given by $C(x) = \sqrt{ax+b}$. where a, b are positive. Using derivatives show that the average and marginal cost curves fall continuously with increasing output.

18. A manufactur's marginal revenue function is given by

    $MR = 275 - x - 0.3x^2$. Find the increase in the manufacturer's total revenue if the production is increased from 10 to 20 units.

ANSWERS

CHECK YOUR PROGRESS 41.1

1. $x = 15, x = 8$
2. $x = 1250$
3. $x = 25, 35$
4. $R(x) = 60x$, $C(x) = 18000 + 15x$, $x = 400$
5. $P(x) = 4500x - 100x^2 - 35000$, $x = 10$, $x = 35$, $x < 10, x > 35$.

CHECK YOUR PROGRESS 41.2

1. (i) $AC = \frac{1000}{x} + 25 + 2x$ (ii) $MC = 25 + 4x$,

   (iii) 85 (iv) 83

2. (i) $AC = 2x + 3 + \frac{4}{x}$ (ii) $MC = 4x + 3$ (iii) 23

3. (i) $AC = 0.002x^2 - 0.04x + 5 + \frac{1500}{x}$

   (ii) $MAC = 0.004x - 0.04 - \frac{1500}{x^2}$

   (iii) $MC = 0.006x^2 - 0.08x + 5$

   (iv) $\frac{d}{dx}(MC) = 0.012x - 0.08$

4. (i) $MC = 0.018x^2 - 0.04x - 30$ (ii) $[MC]_{50} = 13$

5. $x = 6$

6. (i) $R = 25x - \frac{x^2}{4}$ (ii) $AR = 25 - \frac{x}{4}$

   (iii) $MR = 25 - \frac{x}{2}$ (iv) $x = 50, p = 12.5$

7. (i) $x = 2000 - 250p$ (ii) $R = 8x - \frac{x^2}{250}$

   (iii) $AR = 8 - \frac{x}{250}$ (iv) $MR = 8 - \frac{x}{125}$

CHECK YOUR PROGRESS 41.3

1. 300 2. (i) $x=10$ (ii) $x=15$ (iii) $x=15$

3. (i) $x=100$, Rs. 7 4. $x=110$ 5. 40, 400

6. 85 7. 60, 25 8. 4000

CHECK YOUR PROGRESS 41.4

1. $20x-0.02x^2+0.001x^3+7000;\ 20-0.02x+0.001x^2+\frac{7000}{x}$

2. $C=x^3-5x^2+3x+7,\ AC=x^2-5x+3+\frac{7}{x}$

3. $C=4000\sqrt{7x+4}+10000\ ,\ AC=\frac{4000}{x}\sqrt{7x+4}+\frac{10000}{x}$

4. $R=\frac{4x}{6x+9}-x\ ,\ p=\frac{4}{6x+9}-1$

5. $R=20x-\frac{5x^2}{2}+x^3$

6. $p=14-3x+3x^2$

TERMINAL EXERCISE

1. $P(x)=2000x-100x^2-7500;\ x=5,\ 15.$

2. 2300 3. 2000

4. $R(x)=10x\ ,\ C(x)=3x+35000,\ x=350$

5. $x=12, 25$

7. $x>6, 0<x<6,\ C=x^2+5x+36,\ MC=2x+5$

8. $p=8-\frac{x}{250},\ R=8x-\frac{x^2}{250},\ AR=8-\frac{x}{250},\ MR=8-\frac{x}{125}$

9. $x^2+5x+36,\ 2x+5,\ 25$

10. $R=30x+2x^2-5x^3,\ MR=30+4x-15x^2,\ 177$

12. $R=\frac{1}{3}\left(24p-2p^2\right),\ p=6, x=4$

13. (i) $x=10$ (ii) $x=15$ (iii) $x=0, x=15$ 14. 1500, 4500

15. $\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+\frac{116888}{15}$ 18. Rs. 1900

SENIOR SECONDARY COURSE

MATHEMATICS

Questions For Practice - Mathematics for Commerce, Economics and Business

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Maximum Marks : 50 **Time : $1\frac{1}{2}$ Hours**

Instructions :

1. Answer all questions on a separate sheet of paper.
2. Give the following information on your answer sheet :
   - Name
   - Enrolment number
   - Subject
   - Assignment number
   - Address

- Get your assignment checked by the subject teacher at your study centre so that you get positive feedback about your performance.

Do not send your assignment to the National Institute of Open Schooling

1. Find the dividend received annually on 500 shares of par value of 100 each at 5% payable half yearly. (2)
2. Find the income received by investing Rs. 10,000 in 1% stock at 110. (2)
3. The cost function of a firm is given by $C(x) = 2x^2 + 3x + 4$ (2)

   Find the Average Cost (AC).
4. The total cost function of a firm is given as

   $$c(x) = 0.002x^3 - 0.04x^2 + 5x + 1500$$ (2)

   Find the Marginal Cost (MC) function.
5. A man buys 400 shares of a company (of par value Rs. 100 each) for Rs.125 per share and sells them at a premium of Rs. 45 per share. Find his gain. (3)
6. A man bought Rs. 12500 of 8% stock at Rs. 95 and sold it when it rose to Rs. 107. Find his total gain and gain percent (Brokerage : Re 1). (3)
7. The total cost function for a company is given by

   $$c(x) = \frac{3}{4}x^2 - 7x + 27$$

   Find the level of output for which MC=AC (3)

8. If $MR = 20 - 5x + 3x^2$, find the total revenue function. (3)

9. A person invested Rs. 18000 in 8% at 90 and sold the stock at 95 and invested the proceeds in 12% stock, increasing his income by Rs. 680. At what price did he buy the latter stock? (4)

10. Find simple aggregative index number for year 2000 with 1980 as base year

Commodity	A	B	C	D	E
Price(in Rs) in 1980	200	110	20	210	25
Price(in Rs) in 2000	250	150	30	250	25

(4)

11 The demand function for a product marketed by a company is $p = \frac{80 - x}{4}$, where x is the number of units and p is the price per unit. At what value of x will there be maximum revenue? What is this maximum revenue? (4)

12. Using simple average of relatives method, find price index for 2001 taking 1999 as base year.

Commodity	Wheat	Rice	Sugar	Ghee	Tea
Price (per unit in 1999)	10	20	60	40	16
Price (per unit) in 2001	12	22	80	50	20

(6)

13. A person invested Rs. 94649 partly in 8% stock at 104 and remaining in 5% stock at 9%. If his total income from both stocks is Rs. 6240, find the respective investments in each stock. (6)

14. A company sells its product for Rs. 10 per unit. Fixed costs for the company are Rs. 35,000 and the variable costs are estimated to run 30% of the total revenue. (6)

Determine :

(i) the total revenue function.

(ii) total cost function

(iii) quantity the company must sell to cover the fixed cos

QUESTION PAPER DESIGN

Subject : Mathematics **Class : Sr. Secondary level**
Marks : 100 **Duration : 3 Hours**

1. Weightage by Objectives

Objective	Marks	% of the total marks
Knowledge	25	25%
Understanding	50	50%
Application	25	25%

2. Weightage by types of questions

Type	Number of questions	Total marks	% of the total marks	Estimated Time a candidate (in minutes)
Long answer questions	6	36	36%	54
Short answer questions-I	7	28	28%	42
Short answer questions-II	6	18	18%	33
Very short answer questions	9	18	18%	36
Total	**28**	**100**	**100%**	**165 minutes ***

3. Weightage by Content

Sl. no	Units/Sub units (Please specify)	Marks
1.	Algebra	22
2.	Coordinate Geometry	10
3.	Sequences & Series	08
4.	Functions	12
5.	Calculus	26
6.	Statistics	07
7.	Optional Module**	15
	Total	**100**

P.T.O

* It is desired that the learner spends last 15 minutes for revision of the answer script.

**** Optionals are as follows :**

I. Vectors and Three Dimensional Geometry

II. Mathematics for Commerce, Economics and Business

Note: (i) Tabular premium per thousand, mode of payment, rate of rebate ,rate of "No claim Bonus" etc should be mentioned in each question of **optional -I** while framing question paper.

(ii) Rates of taxation like excise/customs duty, surcharge, education cess should be mentioned in each question of **optional -II** while framing question paper.

BLUE PRINT

Subject : Mathematics
Secondary

Class : Senior

Sl. No	Objectives / Form of questions / Content units	Knowledge				Understanding				Application				Total
		E	SA-I	SA-II	VSA	E	SA-I	SA-II	VSA	E	SA-I	SA-II	VSA	
1	**Algebra**	-	4(1)	3(1)	4(2)	6(1)		3(1)**	2(1)	-	-	-	-	**22(7)**
2	**Coordinate Geometry**	-	-	-	2(1)	-	8(2)	-	-	-	-	-	-	**10(3)**
3	**Sequences and Series**	-	-	-	2(1)	-	-	-	-	6(1)	-	-	-	**8(2)**
4	**Functions**	-	-	-	2(1)	6(1)	-	-	-	-	-	-	4(2)**	**12(4)**
5	**Calculus**	-	-	3(1)	-	6(1)	8(2)	-	-	6(1)**	-	3(1)**	-	**26(6)**
6	**Statistics**	-	-	-	-	-	4(1)	3(1)	-	-	-	-	-	**7(2)**
7	**Optional Module** (i) Vectors & Three Dimensional Geometry or (ii) Mathematics for Commerce, Economics and Business	-	-	3(1)	2(1)	-	4(1)	-	-	6(1)				**15(4)**
	Sub-Total	-	4(1)	9(3)	12(6)	18(3)	24(6)	6(2)	2(1)	18(3)	-	3(1)	4(2)	**100(28)**
	TOTAL*	25(10)				50(12)				25(6)				

Note: Figures within the brackets indicate the number of questions and figures outside the brackets indicate marks.

* Denotes that marks have been combined to form one set of question.

*** Internal choice should be given from lessons which have not been covered.*

Summary

Essay	No. 06×6	Marks : 36
Short Answer (SA-I)	No. 07×4	Marks : 28
Short Answer (SA-II)	No. 06×3	Marks : 18
Very Short Answer (VSA)	No. 09×2	Marks : 18

SAMPLE QUESTION PAPER

Subject : Mathematics **Class :** Senior Secondary

Time : 3 Hours **Maximum Marks :** 100

Note : (i) This question paper consists of two Sections, viz., 'A' and 'B'.

(ii) All questions from Section 'A' are to be attempted.

(iii) Section 'B' has got more than one option. Candidates are required to attempt questions from one option only.

SECTION-A

1. If $1, \omega, \omega^2$ are the cube roots of unity, then prove that

$$(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$$ (2)

2. Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ (2)

3. In how many ways can an animal trainer arrange 4 lions and 3 tigers in a row so that no two lions are together. (2)

4. Find the eccentricity and the coordinates of the foci of the ellipse $3x^2 + 4y^2 = 12$ (2)

5. Find the sum of all natural numbers between 100 and 200 which are divisible by 7

OR

The 4th and 9th terms of a G.P. are 8 and 256 respectively. Find the G.P. (2)

6. Let $U = \{1, 2, 3, ..., 10\}$, $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7\}$, verify that $(A \cap B)' = A' \cup B'$ (2)

7. Find the domain of the function $f(x) = \dfrac{1}{\sqrt{(x-1)(2x-5)}}$ (2)

8. Prove that $\tan^{-1}\left(\dfrac{1}{8}\right) + \tan^{-1}\left(\dfrac{1}{5}\right) = \tan^{-1}\left(\dfrac{1}{3}\right)$ (2)

9. If α and β be the roots of the equation

$x^2 - p(x+1) - c = 0$, find the value of $(\alpha+1)(\beta+1)$ (3)

OR

Find the modulus of the complex number $\dfrac{2+i}{3-i}$

10. Using properties of determinants, show that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = bc + ca + ab + abc$$ (3)

OR

Express the matrix $\begin{bmatrix} 2 & -4 \\ 4 & 3 \end{bmatrix}$ as a sum of a symmetric and a skew-symmetric matrix.

11. Find the points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are parallel to x-axis.

OR

Find the intervals in which the function

$$f(x) = x^3 - 9x^2 + 24x + 12 \text{ is}$$

(a) increasing (b) decreasing (3)

12. Evaluate : $\int \sec^3 x \, dx$

OR

Evaluate : $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ (3)

13. Two cards are drawn at random, with replacement, from a well shuffled deck of 52 cards. Find the probability that both the cards are either black or red. (3)

14. Find the middle-term (s) of the expansion

$$\left(x + \frac{1}{2x}\right)^7$$ (4)

15. The portion of a line, intercepted between the coordinate axes, is bisected at the point (2,3). (4) Find its equation.

16. Find the equation of the tangent and normal to the circle

$$x^2 + y^2 - 2x - 10y + 1 = 0 \text{ at the point } (-3, 2)$$ (4)

17. Evaluate : $\lim_{x \to 0} \frac{x^3 \cot x}{1 - \cos x}$

OR

If $y = \sqrt{\frac{\sec x + \tan x}{\sec x - \tan x}}$, find $\frac{dy}{dx}$ (4)

18. Solve the folowing differential equaiton : (4)

$$\frac{dy}{dx} + y \sec x = \tan x$$

19. In a study on effectiveness of a medicine over a group of persons, the following results were obtained.

Percentage of relief	0-20	20-40	40-60	60-80	80-100
No. of patients	10	10	25	15	40

Find the standard deviation of the above data. (4)

20. Using matrix method, solve the following system of linear equations (6)

$$x + 2y + z = 8, 2x - y + z = 3, x + 3y - z = 4$$

21. Find the sum to n terms of the folowing series : (6)

$$\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + ...$$

22. Prove that $\tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha = \cot \alpha$

OR

In any triangle ABC, if a:b:c=7:8:9, show that (6)

$$\cos A : \cos B : \cos C = 14 : 11 : 6$$

23. A right circular cylinder is to be made so that the sum of its radius and its height is 6 m. Find the maximum volume of the cylinder. (6)

24. Evaluate $\int_1^2 x dx$ as limit of a sum. (6)

OR

Find the area enclosed by the circle $x^2 + y^2 = 16$ and x-axis in the first quadrant.

SECTION-B

OPTIONAL-I

(Vectors and Three Dimensional Geometry)

25. Show that the points P,Q and R , with position vectors (2)

$\vec{a} - 2\vec{b}, 2\vec{a} + 3\vec{b}$ and $-7\vec{b}$ respectively are collinear.

26. Find the angle between the vectors (3)

$3\hat{i} + 2\hat{j} - 3\hat{k}$ and $2\hat{i} + 3\hat{j} + 4\hat{k}$

27. Find the equation of the plane through the points $(-1,1,1)$ and $(1,-1,1)$ and perpendicular to the plane $x+2y+2z=5$ (4)

28. Prove that the lines

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7} \quad \text{and} \quad \frac{x-2}{1}=\frac{y-4}{4}=\frac{z-6}{7}$$

are coplanar. Also find the equation of the plane containing these lines. (6)

OR

Find the centre and radius of the circle given by the equations :

$$x^2+y^2+z^2-2y-4z-11=0 \text{ and } x+2y+2z=15$$

OPTIONAL-II

(Mathematics for Commerce, Economics and Business)

25. How much money should be invested to purchase Rs. 9600 stock at 95? (2)

26. A man at the age of 34 years takes a whole life policy (with profits) for Rs. 1 lakh. If the policy covers the risk of accident, calculate the amount of yearly premium assuming the following details: (3)

Tabular premium per thousand	Rs. 30.05
Rebate for large sum assured	Rs. 2 per thousand sum assured
Rebate for mode of payment	3% of tabular premium
Extra premium towards accident benefit	Rs. 1 per thousand

OR

A wholesaler purchases 15 chairs from a the manufacturer at the rate of Rs. 100 per chair excluding tax and sells them to a retailer after adding value of Rs. 50 per chair. Calculate the total tax paid to the Govt. in these transactions by Value Added Tax (VAT) method, if VAT is levied @ 12.5% at each stage.

27. Using simple average of price-relative method, find the price index for the year 2001, taking 1996 as base year from the following data: (4)

Commodity	Wheat	Rice	Sugar	Ghee	Tea
Price(in Rs) in 1996 (per unit)	12	20	12	40	80
Price(in Rs) in 2001 (per unit)	16	25	16	60	96

28. A firm has the following demand and the average cost function: (6)

$$x=480-20p \text{ and } A.C(x)=10+\frac{x}{15}$$

Determine the profit maximizing output and price of the firm.

MARKING SCHEME

(For Sample Question Paper)

Subject : Mathematics **Class : Senior Secondary**

1. $\text{L.H.S.} = (2-\omega)(2-\omega^2)(2-\omega^9.\omega)(2-\omega^9.\omega^2)$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2) \quad [\because \omega^3 = 1]$ 1

$= \left[(2-\omega)(2-\omega^2)\right]^2$ ½

$= \left[4 - 2(\omega^2+\omega) + \omega^3\right]$

$= [4+2+1]^2 = 49 = \text{R.H.S} \quad (\text{Since } 1+\omega+\omega^2 = 0)$ ½

2. $${}^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$ ½

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$ ½

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n+1}{(n-r+1)r}\right]$$ ½

$$= \frac{(n+1)!}{r!(n-r+1)!} = {}^{n+1}C_r$$ ½

3.

L	T	L	T	L	T	L

½

No. of ways of arranging 4 lions in a row = 4! = 24 ½

No: of ways of arranging 3 tigers in a row as shown in the table = 3! = 6 ½

$\therefore$ Required number of arrangements $= 24 \times 6 = 144$ ½

4. The equation of the ellipse can be written as

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

$\Rightarrow \quad a^2 = 4, b^2 = 3$ ½

$b^2 = a^2(1-e^2)$

$\Rightarrow \quad 3 = 4(1-e^2) \Rightarrow e = \frac{1}{2}$ ½

Foci are $(\pm ae, 0)$ i.e. $(1,0), (-1,0)$ 1

5. We have to find the sum of

$105+112+.....+196$ ½

$\therefore \quad 196=105+(n-1).7 \Rightarrow n=14$ ½

$S_{14}=\frac{14}{2}[105+196]=2107$ 1

OR

$T_4=ar^3=8,\ t_9=ar^8=256$

$\therefore \quad \frac{t_9}{t_4}=r^5=32 \Rightarrow \ r=2$ 1

$ar^3=8 \quad \Rightarrow \quad 8a=8 \quad \Rightarrow a=1$ ½

$\therefore$ The required G.P. is 1, 2, 8,....... ½

6. $A \cap B=\phi \qquad \Rightarrow \qquad (A \cap B)'=U$ 1

$A'=\{1,3,5,7,9,10\}$, $B'=\{2,4,6,8,9,10\}$ ½

$\therefore \quad A'\cup B'=U=(A \cap B)'$ ½

7. The given function is $f(x)=\frac{1}{\sqrt{(x-1)(2x-5)}}$

The function is not defined when $(x-1)(2x-5)=0$ ½

For domain of f(x), we have

$(x-1)(2x-5)>0$ ½

$$\therefore \quad \left.\begin{array}{l}(x-1)>0 \text{ and } (2x-5)>0 \\ \text{or} \quad (x-1)<0 \text{ and } (2x-5)<0\end{array}\right\}$$ ½

$\Rightarrow \quad x>\frac{5}{2} \quad \text{or} \quad x<1$ ½

8. $\text{L.H.S.}=\tan^{-1}\left(\frac{1}{8}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\left(\frac{\frac{1}{8}+\frac{1}{5}}{1-\frac{1}{8}\cdot\frac{1}{5}}\right)$ 1

$=\tan^{-1}\left(\frac{\frac{13}{40}}{\frac{39}{40}}\right)$ ½

$$= \tan^{-1}\left(\frac{1}{3}\right) = \text{R.H.S.}$$ ½

9. Here $\alpha+\beta = p$ and $\alpha\beta = -(p+c)$ 1

$$(\alpha+1)(\beta+1) = \alpha\beta + (\alpha+\beta)+1$$ 1

$$\left.\begin{aligned} &= -c-p+p+1 \\ &= 1-c \end{aligned}\right\}$$ 1

OR

$$z = \frac{2+i}{3-i} = \frac{2+i}{3-i} \times \frac{3+i}{3+i} = \frac{5+5i}{10} = \frac{1}{2}+\frac{1}{2}i$$ 1½

$$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$ 1

$$= \frac{1}{\sqrt{2}}$$ ½

10. $$\Delta = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$ 1

$R_1 \to R_1 + R_2 + R_3$

$$\Delta = abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$ 1

$$= (ab+bc+ca+abc)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix} \quad (C_2 = C_2 - C_1, C_3 = C_3 - C_1)$$ ½

$$= (ab + bc + ca + abc)\cdot 1$$

$$= ab + bc + ca + abc$$ ½

OR

$$A = \begin{bmatrix} 2 & -4 \\ 4 & 3 \end{bmatrix} \quad \Rightarrow \quad A' = \begin{bmatrix} 2 & 4 \\ -4 & 3 \end{bmatrix}$$ 1

$$B = \frac{A + A'}{2} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$ ½

$$C = \frac{A - A'}{2} = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$$ ½

$\therefore$ $A = B + C$, where B is symmetric and C is skew-symmetric. 1

11. $$\frac{x^2}{9} + \frac{y^2}{16} = 1 \qquad \text{......(i)}$$

Diferentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-16}{9}\frac{x}{y}$$ 1

When the tangent is parallel to x-axis, $\frac{dy}{dx} = 0$ 1

$\Rightarrow x = 0$

From (i), we get $y = \pm 4$ ½

$\therefore$ The required points are $(0,4),(0,-4)$ ½

OR

$$f(x) = x^3 - 9x^2 + 24x + 12$$

$$f'(x) = 3(x^2 - 6x + 8)$$ ½

For increasing function, $f'(x) > 0$

$$\Rightarrow \quad x^2 - 6x + 8 > 0$$

$$\Rightarrow \quad (x-4)(x-2) > 0 \Rightarrow \left.\begin{array}{r} x > 4 \\ \text{or } x < 2 \end{array}\right\}$$ 1½

$\therefore$ For decreasing function $2 < x < 4$ 1

12. $$I = \int \sec^3 x\, dx = \int \sec x \cdot \sec^2 x\, dx$$

Integrating by parts, we get

$$= \sec x \tan x - \int \tan x \cdot \sec x \tan x\, dx$$ 1

$$= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx$$ ½

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$ ½

$$\left.\begin{array}{l} \therefore \quad 2I = \sec x \tan x + \log(\sec x + \tan x) + c' \\ \therefore \quad I = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log(\sec x + \tan x) + c \end{array}\right\}$$ 1

OR

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \qquad \text{......(i)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx$$ 1

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \qquad \text{.....(ii)}$$ ½

$$\text{(i)} + \text{(ii)} \quad \Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}} dx \cdot 1 = \frac{\pi}{2}$$ 1

$$\Rightarrow \quad I = \frac{\pi}{4}$$ ½

13. P (Black card) $= \frac{26}{52} = \frac{1}{2}$, P (Red card) $= \frac{1}{2}$ 1

P (Both Black cards) $= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ ½

P (Both red cards) $= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ ½

P (Both Black or Both red) $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ 1

14. General term $= t_{r+1} = {}^7C_r \, (x)^{7-r} \left(\frac{1}{2x}\right)^r$ .....(i) 1

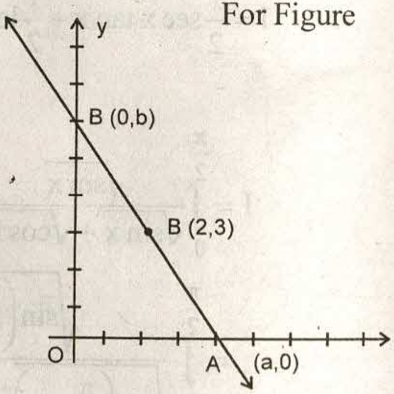
Middle terms are 4th and 5th 1

$\therefore$ Putting r = 3 and r = 4 in (i), we get the middle terms as

$^{7}C_3 (x)^4 \left(\frac{1}{2x}\right)^3$ and $^{7}C_4 (x)^3 \left(\frac{1}{2x}\right)^4$ 1

or $\frac{35}{8}x$ and $\frac{35}{16x}$ 1

15. Let the intercepts on the axes be 'a' and 'b' respectively, For Figure 1



$\therefore$ The equation of line is $\frac{x}{a}+\frac{y}{b}=1$ .....(1) 1

The portion of (1) i.e., AB is bisected at the point (2,3)

$\therefore \quad \frac{a+0}{2}=2 \quad \Rightarrow \quad a=4$

and $\frac{0+b}{2}=3 \quad \Rightarrow \quad b=6$ 1

$\therefore$ Equation of line is $\frac{x}{4}+\frac{y}{6}=1 \Rightarrow \quad 3x+2y=12$ 1

16. The given equation of circle is

$$x^2+y^2-2x-10y+1=0$$

The equation of tangent to circle at (x, y) is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$ 1

Putting $x_1=-3, y_1=2$ and $c=1, g=-1, f=-5$, we have the equation of tangent as ½

$$-3x+2y-(x-3)-5(y+2)+1=0$$

$\Rightarrow \quad 4x+3y+6=0$ .....(i) 1

Equation of normal is $3x-4y+k=0$ ½

It passes through $(-3,2)$

$\therefore \quad -9-8+k=0 \quad \Rightarrow \quad k=17$

$\therefore$ The equation of normal is $3x-4y+17=0$ 1

17. $\lim\limits_{x\to 0}\frac{x^3\cot x}{1-\cos x}=\lim\limits_{x\to 0}\frac{x}{\tan x}\cdot\frac{x^2}{1-\cos x}$ 1

$= \lim\limits_{x\to 0}\frac{x}{\tan x}\cdot\lim\limits_{x\to 0}\frac{x^2}{2\sin^2\frac{x}{2}}$ 1+½

$$1\cdot\lim_{x\to 0}\frac{1}{\frac{2}{4}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2}=2$$ 1+½

OR

$$y=\sqrt{\frac{\sec x+\tan x}{\sec x-\tan x}}\cdot\frac{\sqrt{\sec x+\tan x}}{\sqrt{\sec x+\tan x}}$$ 1

$$=\frac{\sec x+\tan x}{1}=\sec x+\tan x$$ 1

$$\therefore\quad \frac{dy}{dx}=\sec x\tan x+\sec^2 x$$ 1½

$$=\sec x\left[\tan\,x+\sec x\right]$$ ½

18. This is a linear diferential equation.

Here $P=\sec x,\ Q=\tan x$ 1

$$\therefore\quad \text{I.F}=e^{\int P dx}=e^{\int \sec x\,dx}=e^{\log(\sec+\tan x)}=\sec x+\tan x$$ 1

$\therefore$ The solution is

$$y(\sec x+\tan x)=\int \tan x(\sec x+\tan x)dx$$ 1

$$=\int \sec x\tan x\,dx+\int(\sec^2 x-1)\,dx$$

$$=\sec x+\tan-x+c$$ 1

19.

Classes	x_i	f_i	d_i'	f_id_i'	$f_id_i'^2$	
0-20	10	10	−2	−20	40	
20-40	30	10	−1	−10	10	
40-60	50	25	0	0	0	
60-80	70	15	1	15	15	
80-100	90	40	2	80	160	2
		$\Sigma f_i=100$		$\Sigma f_id_i'=65$	$\Sigma f_id_i'^2=225$	

$$\text{S.D}=\frac{C}{N}\sqrt{N\Sigma f_id_i'^2-(\Sigma f_id_i)^2}$$ ½

$$=\frac{20}{100}\sqrt{10\times 225-(65)^2}$$ 1

$$=\frac{1}{5}\sqrt{18275}=27.1$$ ½

20. The given system can be written as

$$AX = B\text{, where } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 3 \\ 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ 1

$$\Rightarrow \quad X = A^{-1}B$$

$$|A| = 11$$ 1

$$\text{Adjoint } A = \begin{bmatrix} -2 & 5 & 3 \\ 3 & -2 & 1 \\ 7 & -1 & -5 \end{bmatrix} \Rightarrow \quad A^{-1} = \frac{1}{11}\begin{bmatrix} -2 & 5 & 3 \\ 3 & -2 & 1 \\ 7 & -1 & -5 \end{bmatrix}$$ 2½

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -2 & 5 & 3 \\ 3 & -2 & 1 \\ 7 & -1 & -5 \end{bmatrix}\begin{bmatrix} 8 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$ 1

$$\Rightarrow \quad x = 1, y = 2, z = 3$$ ½

21. $$t_n = \frac{1}{(2n-1)(2n+1)(2n+3)} = \frac{1}{4}\left[\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n+1)(2n+3)}\right]$$ 2

Putting n = 1, 2, 3,.......

$$\left.\begin{aligned} t_1 &= \frac{1}{4}\left[\frac{1}{3} - \frac{1}{15}\right] \\ t_2 &= \frac{1}{4}\left[\frac{1}{15} - \frac{1}{35}\right] \\ t_3 &= \frac{1}{4}\left[\frac{1}{35} - \frac{1}{63}\right] \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ t_n &= \frac{1}{4}\left[\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n+1)(2n+3)}\right] \end{aligned}\right\}$$ 2

$$\therefore \quad \left.\begin{aligned} S_n &= \frac{1}{4}\left[\frac{1}{3} - \frac{1}{(2n+1)(2n+3)}\right] \\ &= \frac{1}{12} - \frac{1}{4(2n+1)(2n+3)} \end{aligned}\right\}$$ 2

22. $$\tan\alpha - \cot\alpha = \frac{\sin\alpha}{\cos\alpha} - \frac{\cos\alpha}{\sin\alpha} = -\frac{2(\cos^2\alpha - \sin^2\alpha)}{2\cos\alpha\sin\alpha} = -2\cot 2\alpha$$ 2

$$\left.\begin{aligned} \Rightarrow \qquad & \tan\alpha = \cot\alpha - 2\cot 2\alpha \\ \text{Similarly} \qquad & 2\tan 2\alpha = 2\cot 2\alpha - 4\cot 4\alpha \\ & 4\tan 4\alpha = 4\cot 4\alpha - 8\cot 8\alpha \\ & 8\cot 8\alpha = 8\cot 8\alpha \end{aligned}\right\} \quad ....(A)$$ 2

Adding (A), we get

$$\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$$ 2

OR

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{64 + 81 - 49}{144} = \frac{2}{3}$$ 1½

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{81 + 49 - 64}{126} = \frac{11}{21}$$ 1½

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49 + 64 - 81}{112} = \frac{2}{7}$$ 1½

$$\therefore \quad \cos A : \cos B : \cos C = \frac{2}{3} : \frac{11}{21} : \frac{2}{7}$$

$$= 14 : 11 : 6$$ 1½

23. Let r be the radius and b the height of cylinder

$$r + h = 6 \quad \Rightarrow \quad h = 6 - r$$

$$V = f(r) = \pi r^2 (6 - r) = 6\pi r^2 - \pi r^3$$ 2

$$\frac{dV}{dr} = 12\pi r - 3\pi r^2$$ 1

$$\frac{dV}{dr} = 0 \quad \Rightarrow r = 4$$

$$\therefore \quad h = 6 - r = 2$$ 1

Again $\dfrac{d^2V}{dr^2} = 12\pi - 6\pi r$

Showing $\dfrac{d^2V}{dr^2}$ as negative at $r = 4$ 1

$\therefore$ Maximum volume $= \pi \times 16 \times 2 = 32\pi$ 1

24. $I = \int_1^2 x\,dx$, $h = \dfrac{b - a}{n} = \dfrac{1}{n} \Rightarrow nh = 1$ 1

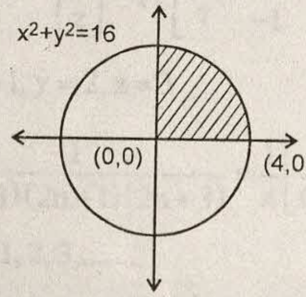
$$\int_1^2 x\,dx = \lim_{h\to 0} h\left[1+(1+h)+(1+2h)+\ldots..+\{1+(n-1)h\}\right]$$ 2

$$= \lim_{h\to 0} h\left[n+h\frac{(n-1)n}{2}\right]$$ 2

$$= \lim_{h\to 0}\left[nh+\frac{(nh-h)\,nh}{2}\right]=\frac{3}{2}$$

$$\therefore \quad \int_1^2 x\,dx = \frac{3}{2}$$ 1

OR



For correct Figure 2

Area of shaded region

$$= \int_0^4 \sqrt{16-x^2}\,dx$$ 2

$$= \left[\frac{x}{2}\sqrt{16-x^2}+8\sin^{-1}\frac{x}{4}\right]_0^4$$

$$= 8\times\frac{\pi}{2} = 4\pi \text{ sq. units}$$ 2

OPTIONAL -I

(Vectors and Three Dimensional Geometry)

25. $\overrightarrow{PQ} = (2\vec{a}+3\vec{b})-(\vec{a}-2\vec{b}) = \vec{a}+5\vec{b}$ ½

$\overrightarrow{QR} = (-7\vec{b})-(2\vec{a}+3\vec{b}) = -2(\vec{a}+5\vec{b})$ ½

$\Rightarrow$ $\overrightarrow{PQ} \parallel \overrightarrow{QR}$ and Q is common ½

$\therefore$ P,Q and R are collinear ½

26. Let θ be the angle between and $\left.\begin{array}{l}\vec{a}=3\hat{i}+2\hat{j}-3\hat{k}\\ \vec{b}=2\hat{i}+3\hat{j}+4\hat{k}\end{array}\right\}$ ½

$$\therefore \quad \cos\theta = \frac{(3\times 2)+(2\times 3)+(-3\times 4)}{\sqrt{3^2+(2)^2+(-3)^2}\cdot\sqrt{2^2+3^2+4^2}} = 0$$ 1½

$\therefore \quad \theta = \frac{\pi}{2}$ 1

27. The equation of plane passing through $(-1,1,1)$ is

$a(x+1)+b(y-1)+c(z-1)=0$ .....(i) 1

This passes through the point $(1,-1,1)$ also

$\therefore \quad 2a-2b+0c=0$ .....(ii) 1

since (i) is prependicular to the plane $x+2y+2z-5=0$

$\therefore \quad a+2b+2c=0$ .....(iii) ½

From (ii) and (iii)

$$\frac{a}{-4}=\frac{b}{-4}=\frac{c}{6} \quad \text{or} \quad \frac{a}{2}=\frac{b}{2}=\frac{c}{-3}$$ 1

$\therefore$ From (i), equation of plane is

$$2(x+1)+2(y-1)-3(z-1)=0$$
$$2x+2y-3z+3=0$$ ½

28. The lines are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$ 1

$$\left.\begin{array}{l} \text{Here, } x_1=-1, y_1=-3, z_1=-5 \\ x_2=2, y_2=4, z_2=6 \\ l_1=3, m_1=5, n_1=7; l_2=1, m_2=4, n_2=7 \end{array}\right\}$$ 1

$$\therefore \quad \begin{vmatrix} 3 & 7 & 11 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix}$$

$$=3(35-28)-7(21-7)+11(12-5)=0$$ 1

$\therefore$ Lines are coplanar.

Equation of the plane containing the lines is

$$a(x+1)+b(y+3)+c(z+5)=0$$ 1

Where $3a+5b+7c=0$

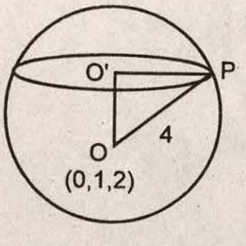
$a+4b+7c=0$ 1

$$\Rightarrow \quad \frac{a}{7}=\frac{b}{-14}=\frac{c}{7} \quad \text{or} \quad \frac{a}{1}=\frac{b}{-2}=\frac{c}{1}$$ ½

∴ The required equation is $(x+1)-2y-6+z+5=0$

$$x-2y+z=0$$ ½

OR



Centre of sphere = $(0,1,2)$

Radius of sphere = OP = 4 1

Equation of plane: $x+2y+2z-15=0$

$$OO'=\left|\frac{0+2+4-15}{3}\right|=3$$ 1

∴ $O'P^2=OP^2=OO'^2=16-9=7 \Rightarrow O'P=\sqrt{7}$ 1

∴ Radius of the circle $=\sqrt{7}$

Equation of line through (0, 1, 2) and prependicular to the plane is

$$\frac{x-0}{1}=\frac{y-1}{2}=\frac{z-2}{2}=k$$ 1

∴ $x=k,\ y=2k+1,\ z=2k+2$

Since it lies on the plane $x+2y+2z=15$

$$k+4k+2+4k+4=15$$

$\Rightarrow$ $k=1$ 1

∴ The coordinates of centre of circle are (1, 3, 4) 1

OPTIONAL -II

(Mathematics for Commerce, Economics of Business)

25. Rs 100 stock can be purchases for Rs. 95 ½

∴ Rs 9600 stock can be purchases for Rs. $\frac{9600\times 95}{100}$ 1

= Rs. 9120 ½

26.

Tabular premium per thousand	= Rs. 30.05	
Rebate for mode of payment = 3% of Rs. 30.05	= Rs. 0.90	½
Rebate for large sum assured	= Rs. 2.00 per thousand sum assured	
Total = Rs. $(30.05-0.90-2)$	= Rs. 27.15	½
Extra premium towards accident benefit	= Rs. 1.0 per thousand	
Annual premium per thousand	= Rs. 28.15	1

$\therefore$ Annual premium to be paid $= \text{Rs. } \frac{28.15}{1000} \times 100000 = \text{Rs. } 2815.00$ 1

OR

Wholesalers cost price $= \text{Rs. } (15 \times 100) = \text{Rs. } 1500$

VAT @ 12.5% $= \text{Rs. } 1500 \times \frac{12.5}{100} = \text{Rs. } 187.50 \approx \text{Rs. } 188$

Total = Rs. 1688 1

Value added @ Rs. 50 per chair = Rs. 750

Total = Rs. 2438 ½

VAT on the added amount $= \text{Rs. } 750 \times \frac{12.5}{100} = \text{Rs. } 93.75 \approx \text{Rs. } 94$ 1

Therefore, final selling price is Rs. (2438 + 94) = Rs. 2532

Total VAT paid to the Govt. is Rs. (188 + 94) = Rs. 282. ½

27.

Commodity	Price in 1996 (P_0)	Price in 2001 (P_1)	$\frac{P_1}{P_0} \times 100$
Wheat	12	16	133.33
Rice	20	25	125.00
Suger	12	16	133.33
Ghee	40	60	150.00
Tea	80	96	120.00
			Total 661.66

3

$\therefore$ Price index for 2001 taking 1996 as base year $= \frac{661.66}{5}$

$= 132.33$ 1

28. $20p = 480 - x$

$\Rightarrow \quad p = 24 - \frac{x}{20}$ ½

$R(x) = p.x = 24x - \frac{x^2}{20}$ .....(i) 1

$AC(x) = 10 + \frac{x}{15}$

$C(x) = 10x + \frac{x^2}{15}$ .....(ii) 1

$$P(x) = (i) - (ii) = 14x - \frac{7x^2}{60}$$ 1

$$\Rightarrow \qquad P'(x) = 14 - \frac{7x}{30}$$

For maximum profit $P'(x) = 0 \Rightarrow x = 60$ 1

$$P''(x) = -\frac{7}{30} \Rightarrow \text{maximum}$$ ½

$$\therefore \qquad p = 24 - \frac{60}{20} = 21$$

$\therefore$ Maximum output $= 60$

and price at this output $= 21$ 1

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		Easy	Difficult	Interesting	Confusing	Simple	Complex	Useful	Not useful	Very helpful	Somewhat helpful	Not helpful
32												
33												
34												
35												
36												
Module-2												
37												
38												
39												
40												
41												

Fourth fold

Third fold

Feed back on Questions

Lesson no.	Lesson Name	Intext Questions		Terminal Question		
		Useful	Not useful	Easy	Diff.	V. diff.
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35						
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Module-2						
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FUNDAMENTAL DUTIES

Part IVA (Article 51 A)

It shall be the duty of every citizen of India-

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forest, lake, rivers and wild life, and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that nation constantly rises to higher levels of endeavour and achievement.

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